MECHANICS
FOR
ENGINEERS

Part I
STATICS & HYDROSTATICS

By

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PREFACE TO THE FIRST EDITION

This book has been written to meet the requirements of the degree students of engineering and technology. The aim of the Book is to give to the students a clear understanding of the fundamental principles of mechanics and to enable them to apply these principles to problems in engineering. To achieve this purpose, simple examples have been given based on common experience as well as engineering practice. More stress is laid on numerical examples than is usually given in the books for non-technical students. Complicated problems in engineering, however, have been avoided. It is felt that such problems would be better tackled in their place of occurrence, once the student gets a sound grasp of the fundamentals by trying his hand on simplified problems.

The book has been divided into three parts : Statics, Hydrostatics and Dynamics. Besides the usual topics, chapters on Graphical Methods, Elasticity (including bending of beams), and Forces in Space are given in Statics to make the book more useful to the students of engineering. A knowledge of Calculus is essential for the study of Dynamics, as the definitions of velocity and acceleration in this book have been based on a concept of limits. The examples are ample in number and are well graded. Of these a large number are original, many have been taken from the examination papers of various engineering institutions and universities, and the others are taken from various text-books on the subject. Reference in a problem to some institution is not meant to indicate the source of the problem, but merely to indicate the type of questions asked in the engineering examinations of that institution.

Our thanks are due to Dr. Gorakh Prasad, Dr. Pran Nath, Dr. Hira Lal Agarwal, Dr. Shanti Ram Mukherji for the help given in the writing of this book.

DECEMBER, 1960

CHANDRIKA PRASAD
RAJ KRISHNA BHATT

PREFACE TO THE SECOND EDITION

Besides extensive revisions of sections on General Principles

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Part I—Statics & Hydrostatics

MECHANICS FOR ENGINEERS

PART I—STATICS

CHAPTER I

GENERAL PRINCIPLES & DEFINITIONS

1·1. Introduction. Mechanics is the branch of physical science which deals with the effect of forces upon bodies in motion or at rest. The subject of mechanics is generally divided into two parts : statics and dynamics. Statics is the branch of mechanics concerned with the conditions under which the forces acting on a body are so balanced as to cause no change in its state of rest, and an equilibrium exists. Dynamics is the branch dealing with the motion of bodies.

Mechanics plays an important part in engineering. Its importance lies in the fact that the design of structures and machinery is governed by the laws of mechanics. In applying the principles of mechanics to engineering, the problems are often simplified down to retain only the salient features and to make the practical analysis simpler. For example, to obtain the forces in the members of a steel bridge the girders may be considered to be thin rods pin-jointed at their ends.

To develop the basic laws of mechanics we introduce certain mathematical idealisations. We consider a *particle* to be a small portion of matter occupying a definite position but having no dimensions. A body is a part of matter limited in every direction. It occupies some definite space and has a definite size and shape. It may be considered as a collection of particles joined together. A body is said to be *rigid*, when the relative distances of its particles remain fixed and do not change on the application of forces. Physical bodies are not

small that they may be disregarded and the body may be taken as rigid.

1·2. Fundamental Quantities and Units. In the development of the laws of mechanics, certain concepts are considered fundamental. They are : time, space, mass and force. The student is expected to have an intutive knowledge of these as no satisfactory definitions can be given.

Time is recognised by succession of events. *Space* is recognised as extension in all directions. The position of a point in space is generally determined by measuring its distance from a fixed standard point in three mutually perpendicular directions. *Mass* of a body is the quantity of material (or matter) in it. Masses of bodies are proportional to the gravitational pulls exerted on them by the Earth. Hence the mass of a body is usually determined by means of the lever-arm balance.

Two systems of units are used in the measurement of mass and distance. In the metric system the unit of mass is the gram (gm.*) and the unit of distance is the centimetre (cm.). In the British system the units of mass and distance are respectively the pound (lb.) and the foot (ft.). The unit of time in both the systems is the same, namely the second (sec.).

The gram and the centimetre are defined in terms of pieces of metal preserved in Paris, and the pound and the foot in terms of pieces of metal preserved in London. By an act of parliament passed in 1956, the metric system has become the recognised system of units in India. Often the units gram and centimetre prove too small for engineering practice. Larger units are kilogram (kg.) and metric tonne, or simply tonne(t.), for mass and metre (m.) and kilometre (km.) for distance. The conversion relations are

$$1000 \text{ gm.} = 1 \text{ kg.}, \quad 1000 \text{ kg.} = 1 \text{ tonne.}$$

$$100 \text{ cm.} = 1 \text{ m.}, \quad 1000 \text{ m.} = 1 \text{ km.}$$

Force is recognised by the push or pull that changes or tends to change the motion of a body or its state of rest.

*The abbreviation g. is also used for gram, but we shall use gm. to avoid confusion with g, the acceleration due to gravity.

Our primary concept of force arises from the muscular effort required to push or pull an object. In the present mechanical age this push or pull may be supplied by the steam power or by the explosive combustion of petrol. Another commonly experienced force is the Earth's pull on a body. This pull of the Earth on the body is called its *weight* and can be measured by a spring balance. The weight of a body varies slightly from place to place, but the variation is so small that it may be neglected in practically all the problems in engineering.

Forces always occur in pairs, so that if there is a force exerted by the first body on the second body, there is an equal and opposite force exerted by the second body on the first body. This fact was stated by Newton in his third law of motion. But in dealing with the equilibrium of one body, usually one force of the pair enters into consideration.

Forces are of various kinds. Some arise on account of universal laws of nature. For example, each particle in nature attracts another particle with a force proportional to the product of the masses of the two particles and inversely proportional to the square of the distance between them. This law by Newton embodies the gravitational property of matter. The weight of a body arises on this account. The Earth exerts a gravitational pull on each particle of the body. The pulls on the various particles can be combined into a single force acting through a point of the body known as its centre of gravity. Other forces of a similar nature are the attractive or repulsive forces between electrically charged particles, or between magnetic particles.

Some forces arise because of mutual interaction between the particles of a body. For example when the two ends of a string are pulled apart, the particles of the string resist it by pulling each other together. This causes a force of *tension* in the string. Tension will also arise in a spring or a rod under similar conditions. If the two ends of a spring or a rod are pushed in then a force of compression will arise.

A third type of force arises when a body presses against a surface. The surface will press the body back with a force known as *reaction*. For a smooth surface the reaction acts in a direction normal to the surface. For rough surfaces a second force, known as *friction*, acting parallel to the surface, also comes into play.

When a body is hinged or supported at a point, then also a reactive force acts upon the body at the hinge or the support. Besides these types of forces, we may have forces due to the pressure of the wind or water on a surface.

In metric system, the unit of force commonly used in statics is kilogram's weight (kg. wt.), the force exerted on a mass of one kilogram by the Earth. Frequently the word 'weight' is dropped from the name of the unit, and we say 'a force of one kilogram'. Other units of force are gram's weight, tonne's weight, and in the British system, pound's weight (lb. wt.).

We shall see in Part II that still other units of force can be defined from Newton's second law of motion. These are the dyne, newton (N.) and poundal. The conversion relations for these are

$$1 \text{ gm. wt.} = 980{\cdot}7 \text{ dynes.} \quad 1 \text{ kg. wt.} = 9{\cdot}807 \text{ N.}$$

$$1 \text{ N.} = 10^5 \text{ dynes.} \quad 1 \text{ lb. wt.} = 32{\cdot}17 \text{ poundals.}$$

The factors 980·7 and 32·17 occurring in the above relations are the mean values of g (see Part II) in metric and British systems. In application to practical problems they are generally rounded off to the figures 980 and 32 respectively.

1·3. Vectors. In order to completely describe a force, it is necessary to know (i) its magnitude, (ii) its point of application, and (iii) its direction. These three properties of a force are called its *elements* or *characteristics*. A force differs in this respect from quantities like length or mass which are characterised by their magnitudes only.

Quantities which possess magnitude only are called *scalar* quantities, e.g., time, mass, volume. Quantities which have magnitude as well as direction are called *vector* quantities, e.g., force, velocity, acceleration.

A vector quantity is graphically represented by a segment of a straight line drawn parallel to the vector. The sense is represented by an arrow-head on the line, and the magnitude of the vector is represented by the length of the line according to some convenient scale.

Vectors are of two types. When a vector can be represented by any one of the parallel directed segments and its position its immaterial, it is called a *free vector*. When the position matters, the vector represented by

the directed segment through the point of application is called the *localised* or *bound* vector.

1·4. Resultant of two forces. When two or more forces act on a rigid body; it is often possible to replace them by a single force whose effect on the body is same as the combined effect of all the forces acting on the body. This force, which is equivalent to all the other forces, is known as their *resultant*. The resultant of two forces is given by the following *law of parallelogram of forces*:

If two forces P and Q acting at a point A are represented by the vectors $\overline{AB}$ and $\overline{AC}$, their resultant R is represented by the diagonal $\overline{AD}$ of the parallelogram constructed with vectors $\overline{AB}$ and $\overline{AC}$ as its sides.

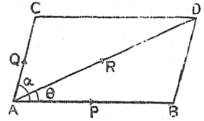


Fig. 1

Thus, in vector notation,

$$\overline{AB}+\overline{AC}=\overline{AD}, \qquad (1)$$

which is the law for addition of any two vectors.

This is the fundamental law on which many of the operations of mechanics are based. No satisfactory mathematical proof can be offered, but the correctness of the law can be verified experimentally. When the two forces act in the same direction, the law reduces to simple algebraic addition.

The law of parallelogram of forces can also be stated in the following alternative form:

The triangle law. If two forces acting at a point are represented by the two sides $\overline{AB}$, $\overline{BD}$ of a triangle, then the third side $\overline{AD}$ represents their resultant both in magnitude and direction.

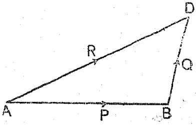


Fig. 2

Thus from the triangle of forces ABD,

$$\overline{AB}+\overline{BD}=\overline{AD}.$$

This follows from (1) since $\overline{BD}=\overline{AC}$. It should be noted

however, that whereas the vectors in figure 1 are localised vectors, those in figure 2 are free vectors. The resultant acts through the same point at which the component forces are acting. The triangle of forces is more convenient to apply graphically.

These laws may be used either for graphic or for algebraic solution of a problem. In the graphic solution, an accurate drawing is made to a definite scale and the resultant is computed from the drawing by measurement.

In the algebraic solution of a problem trigonometrical relations are used to obtain the resultant. Thus from fig. 1 we have,

$$AD^2 = AB^2 + BD^2 - 2\,AB\,.\,BD \cos ABD$$

$$= AB^2 + AC^2 - 2\,AB\,.\,AC \cos (180^\circ - BAC),$$

or

$$R^2 = P^2 + Q^2 + 2PQ \cos \alpha,$$

where α is the angle between P and Q. This gives the magnitude of the resultant.

For the direction of the resultant, we have from $\triangle ABD$ (fig. 1)

$$\frac{\sin ABD}{AD} = \frac{\sin DAB}{BD} = \frac{\sin ADB}{AB},$$

or

$$\frac{\sin (180 - \alpha)}{R} = \frac{\sin \theta}{Q},$$

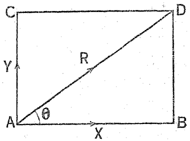
or

$$\sin \theta = \frac{Q \sin \alpha}{R}.$$

1·5. Resolution of a force into components. By reversing the parallelogram or triangle law a force can be resolved into two components; that is the action of one force can be replaced by that of two forces which will produce the same effect as the given force. This process is called *resolution*. Thus the force R in fig. 2, page 5, can be resolved into components P and Q, both acting at the point A.

Since the point B in fig. 2 can be chosen at random, hence a given force can be resolved into two components in an infinite number of ways. The resolution of a force into two components which are mutually perpendicular to one another, called *rectangular components*, is of special importance.

Suppose the force R is to be resolved into two components one making an angle θ with R and the other at right angles to the first. Completing the rectangle $ABDC$, we see that the two components X and Y are given by

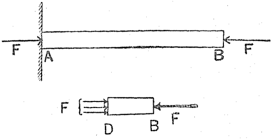


$$X = R \cos\theta,$$

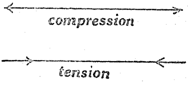
and $$Y = R \sin\theta.$$

1·6. Equilibrium of two forces. When the resultant of a system of forces acting on a body vanishes, the forces are said to balance and the system is in *equilibrium.* It follows that a body cannot be in equilibrium under the action of a single force.

When two forces act at a point of a body, they will have a non-zero resultant except when the forces are equal in magnitude and opposite in direction. When the forces are equal and opposite, the body will be in equilibrium. Now consider two equal and opposite forces having the same line of action but acting at two different points of the body. For example, when a rod AB is pressed normally against a smooth wall by a force F acting at B in the direction BA, an equal and opposite reaction F is exerted by the wall which



Forces exerted by the body in compression/tension



acts on the rod at A in the direction AB. These two forces at A and B try to effect a compression of the rod. If the rod is rigid no change in length will occur, but internal forces will be set up beween the particles opposing the compression. If we cut the rod at a point D, the forces exerted by the part AD on the part DB will be as shown in the second figure. The forces exerted by any part CD of the rod (or by the whole rod AB) on the rest of the system are indicated by arrows in the third figure. A similar state of affairs will exist for a body pulled apart by two forces acting at two different points.

Hence *two equal and opposite forces acting on a rigid body at two different points will keep it in equilibrium if they are* COLLINEAR. In engineering practice a rod in compression under two forces is called a *strut*, and a rod in tension in called a *tie*.

1·7. Transmissibility of forces. We have seen above that when two forces equal in magnitude and opposite in direction act on a rigid body along the same line, they balance one another. A consequence of this is that *when a force acts upon a rigid body its effect is unchanged if it be transferred to any other point on the line of action of the force.*

Thus a force F acting at a point A on a body can be transferred to any other point, say B, on the line of action of the force.

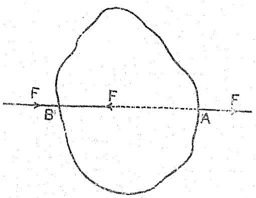


Fig. 5

PROOF. Introduce at B two equal and opposite forces F and $-F$. This will not disturb the condition of the body. Now the force F at A and $-F$ at B will annul each other and a force F at B will be left. This proves the principle of transmissibility of forces.

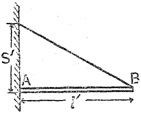
The student should note that the proposition may **not** hold for bodies which are not rigid, since the force F at A and $-F$ at B may change the dimensions of a non-rigid body.

1·8. Space diagram and free-body diagram. A problem in statics generally consists of a statement of

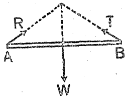
certain known and unknown quantities and of relations between them, from which some unknown elements are required to be determined. For example, suppose it is given that one end of a rod, of weight W pounds and length l feet, is pinned to a wall and is kept horizontal by a cord attached to the other end, the free end of the cord being fastened to a point on the wall, s feet vertically above the pinned end; and it is required to find the tension in the cord.

The following procedure will be useful to analyse the problem :

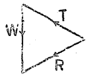
First draw a *space diagram* to show the situation of the body as it exists, including surrounding parts [fig. (*a*)].



(a) Space diagram



(b) Free-body diagram



(c) Force diagram

Next draw a *free-body* diagram for the body separated from the surrounding parts, with the forces acting on the body due to these parts (and any other forces) shown by localised vectors [fig. (*b*)]. Quite often it is not necessary to draw a separate diagram, as in figure (*b*), but having drawn the space diagram, that very figure is turned into a free-body diagram by drawing directed lines to indicate the forces.

In the present case there are three forces acting on the body. The tension T of the string, the weight W of the body, and the reaction R of the pin. The tension acts along the string. The weight of the body acts vertically through the centre of gravity of the body. The third force is drawn to pass through the intersection of the other two, since, as we shall see in chapter II, three forces in equilibrium meet at a point.

When there are several bodies jointed together separate free-body diagrams may be drawn for each part. From the free-body diagram we can draw a *force diagram* [fig. (*c*)]. In this diagram, the forces acting on the body are represented (on any particular scale) by free vectors drawn end to end in order.

Ex. 1. A man of weight W stands on a rung two-thirds way up a step-ladder placed on smooth level ground. The weight of each arm of the ladder is w and its length $2a$. The two arms are connected at their middle by a weightless rod of length a. Draw free-body diagrams for the two arms of the ladder and the tie rod.

The space diagram is shown in fig. 1 and the free-body diagrams in fig. 2.

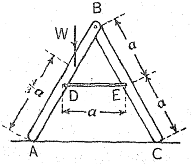


Fig. 1

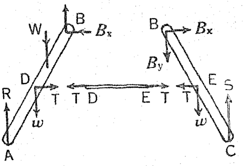


Fig. 2

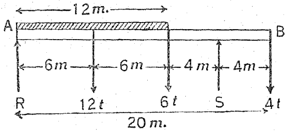
To draw the free-body diagram for arm AB, we notice that the reaction R of the ground at A will be vertically upwards, being normal to the ground; the tension T of the tie rod at D will be in the direction DE; the weight w of the ladder will act at D and the weight W will act at $\frac{4}{3}a$ from A towards B, both vertically downwards. The reaction of the hinge at B is not known in magnitude or direction. So we show it by its two components B_x and B_y.

Similarly for BC the reaction S at C is vertically upwards, the tension at E is in direction ED and the weight w at E is vertically downwards. Since by Newton's third law the action and reaction are equal and opposite, the reaction of the hinge B on BC has the same components B_x and B_y but in directions opposite to those for the arm AB.

The free-body diagram for the tie rod DE consists simply of two equal and opposite forces acting at D and E. Again applying Newton's third law we notice that the forces at D and E, whether acting on AB, BC or DE, have all the same magnitude T.

Ex. 2. A beam AB, 20 metres long, is simply supported at A and a point 4 metres from B. It carries a uniformly distributed load of 1 tonne per metre for the first 12 metres of its length, a point load of 6 tonnes at 12 metres from A, and a second point load of 4 tonnes at B. Draw the free-body diagram for the beam.

In this case we draw the space diagram first and turn that diagram into a free-body diagram by indicating the forces. All the weights and the reactions are vertical forces, the former acting downwards and the latter upwards at the specified points. The distributed load can be taken as a single force of 12 tonnes acting through the middle point of the 12 metre span. The completed diagram will be as shown.



1·9. Methods of solution of problems. There are two common methods for the solution of problems in statics:

(i) Graphical method,
(ii) Analytical method.

In *graphical method*, the forces, known from the free-body diagram, are plotted to scale and then the unknown elements are found from the completed force diagram.

In *analytical method*, the unknown forces are represented by letters in the free-body diagram, the equations of equilibrium are framed and then solved.

In order to understand thoroughly a subject such as applied mechanics, it is necessary for a student to know both these methods and to solve a good number of problems by both the methods. If time permits, solution by one method and a check solution by the other method will give confidence in the results obtained.

Examples 1

1. A body of weight 750 kg. lies on a plane inclined at 27° with the horizontal. Resolve the force parallel to the line of greatest slope, and perpendicular to the plane.

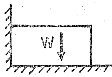
2. Resolve a force of 50 newtons into components whose sum shall be 75 N. and which shall be inclined at an angle of 120°.

3. Show that the resultant of two forces, acting at a point O in the direction OA and OB and represented in magnitudes by $\lambda . OA$ and $\mu . OB$, is represented by $(\lambda + \mu) . OC$, where C is a point in AB such that $\lambda . CA = \mu . CB$.

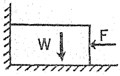
4. The resultant of two forces P and Q is of magnitude P. Show that if the force P be doubled, Q remaining the same, then the new resultant will be perpendicular to Q and its magnitude will be $\sqrt{(4P^2 - Q^2)}$.

5. ABC is a triangle, and D, E and F are mid-points of sides BC, CA and AB respectively. Forces represented by $\frac{2}{3}\,BE$ and $\frac{1}{3}\,CF$ act at a point where AD and BE meet. Show that the resultant is represented in magnitude and direction by $\frac{1}{2}AB$ and that the line of action divides BC in the ratio 2 : 1.

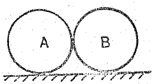
[*Banaras*, '32]



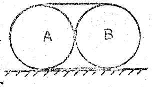
6. Does a contact between two bodies always give rise to an action and reaction between them? Indicate the reactive forces on the block in the two cases shown in the adjoining figures.



7. Two equal cylinders A and B each of weight W, are placed on a horizontal plane in contact with one another. Is there any action and reaction between them at the point of contact?



If an endless stretched rubber band holds them together, draw the free-body diagrams of (i) the cylinder A, (ii) the two cylinders combined.



8. A block of weight W_1 is placed on a block of weight W_2 which is placed on a horizontal table (Fig. 8). Draw the free-body diagrams of the two blocks separately, and of the two blocks combined.

9. Draw the free-body diagram of a heavy ball held by a string and in contact with a smooth vertical wall (Fig. 9).

10. A tie rod AB supports a load W (Fig. 10). Draw the free-body diagram of the portion of the rod lying below CD.

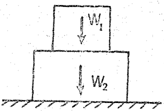


Fig. 8

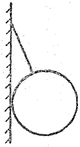


Fig. 9

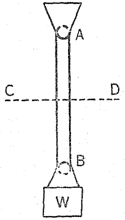


Fig. 10

11. Two smooth spheres of weights W_1 and W_2 rest within a smooth cylinder of weight W which is placed on a horizontal table (Fig. 11). Draw separate free-body diagrams of the two spheres and the cylinder. Assume the radii of the spheres and the cylinder to be r_1, r_2 and r respectively.

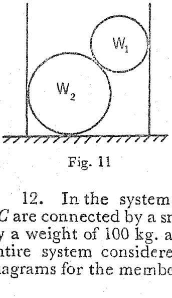


Fig. 11

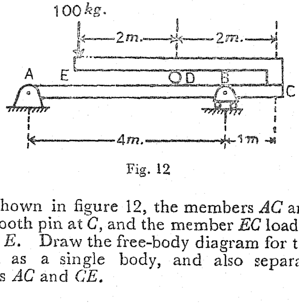


Fig. 12

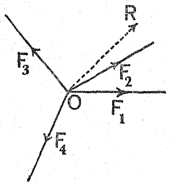
12. In the system shown in figure 12, the members AC and EC are connected by a smooth pin at C, and the member EC loaded by a weight of 100 kg. at E. Draw the free-body diagram for the entire system considered as a single body, and also separate diagrams for the members AC and CE.

CHAPTER II

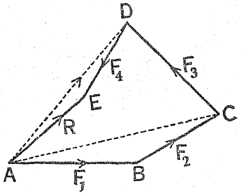
COPLANAR CONCURRENT FORCES

2·1. Resultant of coplanar concurrent forces.

(i) *The Graphical Method.* Suppose the forces F_1, F_2, F_3 and F_4 act at a point O, as in figure (a) below. Then using the triangle law their resultant may be found as follows.



(a) Free-body diagram



(b) Force diagram

From any point A, draw free vectors $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DE}$ to some convenient scale parallel to the forces F_1, F_2, F_3 and F_4 to represent them in order [fig. (b)]. The resultant R is then given both in magnitude and direction by the closing line AE of the force polygon $ABCDE$.

Proof. Join AC, AD and AE.

From the $\triangle ABC$: $\overline{AC}=\overline{AB}+\overline{BC}$.

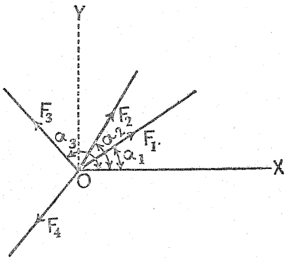
" " $\triangle ACD$: $\overline{AD}=\overline{AC}+\overline{CD}=\overline{AB}+\overline{BC}+\overline{CD}$.

" " $\triangle ADE$: $\overline{AE}=\overline{AD}+\overline{DE}=\overline{AB}+\overline{BC}+\overline{CD}+\overline{DE}$.

Hence $\overline{AE}$ is the resultant of the given forces F_1, F_2, F_3 and F_4.

(ii) *The Analytical Method.* Let $F_1, F_2, F_3, \ldots$ be a system of coplanar forces acting at a point O.

Take a set of rectangular axes OX and OY, and let $F_1, F_2, F_3, \ldots$ make angles $\alpha_1, \alpha_2, \alpha_3, \ldots$ with OX.



Now, the force F_1 can be replaced by the rectangular components $F_1 \cos \alpha_1$ along OX and $F_1 \sin \alpha_1$ along OY. Similarly, the forces $F_2, F_3, \ldots$ can be replaced by the components $F_2 \cos \alpha_2, F_3 \cos \alpha_3, \ldots$ along OX and $F_2 \sin \alpha_2, F_3 \sin \alpha_3, \ldots$ along OY.

The components along OX can be added up to a single force X, and those along OY can be added up to a force Y, where

$$X = F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + F_3 \cos \alpha_3 + \ldots,$$

and $$Y = F_1 \sin \alpha_1 + F_2 \sin \alpha_2 + F_3 \sin \alpha_3 + \ldots.$$

Thus the system reduces to two forces X and Y along the two axes. These can be combined to give a single resultant R acting through O at an angle θ to the x-axis. By § 1·4 we see that the magnitude of the resultant R is given by

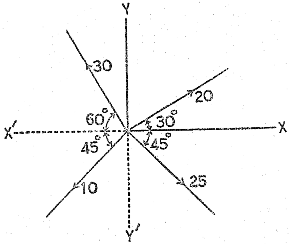
$$\sqrt{(X^2+Y^2)},$$

and the direction by

$$\tan \theta = Y/X.$$

Ex. Find the resultant of the four forces represented in the figure on the next page. The forces are given in kilograms.

Let R be the resultant of the system, acting at an angle θ with the x-axis. Then



$$R\cos\theta = X = 20\cos 30° + 30\cos(180°-60°) + 10\cos(180°+45°) + 25\cos(360°-45°)$$
$$= 20(\sqrt{3}/2) - 30\times\tfrac{1}{2} - 10(1/\sqrt{2}) + 25(1/\sqrt{2})$$
$$= 12{\cdot}92 \text{ kg.}$$

$$R\sin\theta = Y = 20\sin 30° + 30\sin 120° + 10\sin 225° + 25\sin 315°$$
$$= 20\times\tfrac{1}{2} + 30(\sqrt{3}/2) - 10(1/\sqrt{2}) - 25(1/\sqrt{2})$$
$$= 11{\cdot}24 \text{ kg.}$$

$\therefore$ $R = X^2 + Y^2 = \sqrt{\{(12{\cdot}92)^2 + (11{\cdot}24)^2\}} = 17{\cdot}1$ kg.

and $\tan\theta = Y/X = 11{\cdot}24/12{\cdot}92 = 0{\cdot}87$.

$\therefore$ $\theta = 41°$ in the first quadrant, since X and Y are both positive.

2·2. Moment of a force. The *moment of a force* about a point is defined as the product of the force and the perpendicular distance of its line of action from the point.

Let F be the force, O the given point, and p the perpendicular distance of the line of action of the force F from O. Then the moment of the force F about O is Fp.

The perpendicular distance p is called the *arm of the force* and the point O about which the moment is taken, is called the *centre of moment*.

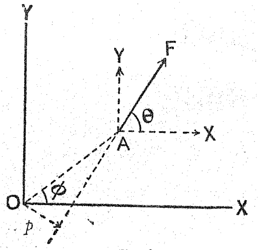


Fig. 1

Let X, Y be the components of the force F and let θ be the angle it makes with x-axis. Then

$$X = F\cos\theta,\ Y = F\sin\theta.$$

Let (x, y) be the coordinates of a point A on the line of action of the force F, and let ϕ be the angle OA makes with the x-axis. The moment of force F about O

$$\begin{aligned}
&= Fp \\
&= F\,.\,OA\sin(\theta-\phi) \\
&= F\,.\,OA(\sin\theta\cos\phi-\cos\theta\sin\phi) \\
&= F\sin\theta\,.\,OA\cos\phi - F\cos\theta\,.\,OA\sin\phi \\
&= Yx - Xy. \qquad \ldots \quad (1)
\end{aligned}$$

Let $KLMN$ be a plane containing the force F and the centre of moment O (fig. 2). It will be seen that the moment of the force about O measures the *turning effect* of the force F about an axis AOB, which is normal to the plane containing the force F and the point O. This normal line through O, is called the *axis of the moment.*

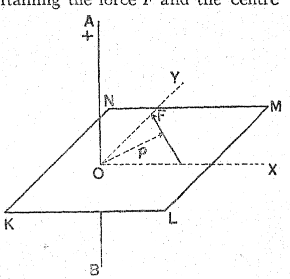


Fig. 2

The units for measuring the moment of a force are gramme-centimetre (gm-cm.), kilogramme-metre (kg-m.), or pound-foot (lb-ft.). A moment which produces anti-clockwise turning effect, when viewed from the positive side of the axis, is regarded as a *positive moment*, and a clockwise turning effect is taken as a *negative* moment. Thus the moment of the force F in figures 1 and 2 is positive, while the moment of F in figure 3 is negative. For the last case, the moment of F about O

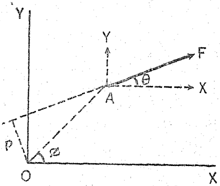


Fig. 3

$$\begin{aligned}
&= -Fp = -F.\ OA\sin(\phi-\theta) \\
&= F.OA\sin(\theta-\phi) \\
&= Yx - Xy, \text{ as in } (1).
\end{aligned}$$

The moment of a force about a point is a vector quantity acting along the axis of the moment. When a number of forces

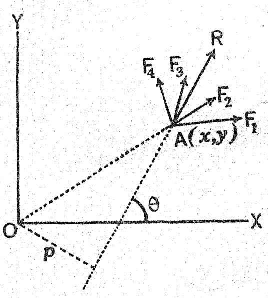
act in a plane, their moments about a point O in the plane are various vectors all acting along the same axis. So the resultant moment (total turning effect) due to all the forces is the algebraic sum of the moments of the forces about O. We shall now show that this resultant moment is equal to the moment about O of the resultant of all the forces.

2·21. Principle of Moments. *The algebraic sum of the moments of a system of coplanar concurrent forces about any point in their plane is equal to the moment of their resultant about the same point.*

Let $F_1, F_2, F_3, \ldots$ be a system of coplanar concurrent forces at $A\,(x, y)$ and let O be the centre of moments. Let $X_1,\ Y_1;\ X_2,\ Y_2;\ X_3,\ Y_3;\ \ldots$ be the components of the forces $F_1, F_2, F_3, \ldots$ parallel to the axes, and X, Y those of R, the resultant of the system. Let $p_1, p_2, p_3, \ldots\ p$ be the lengths of perpendiculars from O on $F_1, F_2, F_3, \ldots, R$ respectively. Then, we have,

$$X = X_1 + X_2 + X_3 + \ldots,$$

$$Y = Y_1 + Y_2 + Y_3 + \ldots .$$



By equation (1), §2·2, the sum of the moments of the forces about O

$$= F_1p_1 + F_2p_2 + F_3p_3 + \ldots$$

$$= (xY_1 - yX_1) + (xY_2 - yX_2) + (xY_3 - yX_3) + \ldots$$

$$= x(Y_1 + Y_2 + Y_3 + \ldots) - y(X_1 + X_2 + X_3 + \ldots)$$

$$= xY - yX = Rp.$$

Hence the sum of the moments of the forces about O is equal to the moment of the resultant about the same point.

EXAMPLES 2

1. Forces P and Q cut at a point O and their resultant is R; if any transversal cut the forces in the points L, M, N respectively, show that

$$\frac{P}{OL}+\frac{Q}{OM}=\frac{R}{ON}.$$ [*Jodhpur*, 1965]

2. The resultant of forces P and Q is R; if Q is doubled, R is doubled whilst if Q is reversed, R is again doubled. Show that $P : Q : R : : \sqrt{2} : \sqrt{3} : \sqrt{2}$. [*Ranchi*, 1965]

3. If forces of magnitudes P, Q and R act at a point, parallel to and in the same order as the sides of an equilateral triangle, prove that the magnitude of the resultant is

$$\sqrt{(P^2+Q^2+R^2-PQ-PR-QR)}.$$

4. Three forces P, Q and R act at a point in a plane. The angles between, P and Q, and Q and R, are α and β respectively. Show that the magnitude of the resultant is

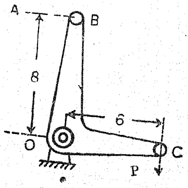
$$\sqrt{\{P^2+Q^2+R^2+2PQ\cos\alpha+2QR\cos\beta+2PR\cos(\alpha+\beta)\}}.$$

5. Determine the magnitude and the direction of the resultant of the forces acting at a point and having the magnitudes and directions given in the following table.

(*a*)					
	Forces in kg. wt.	100	80	50	70
	Angles	20°	60°	135°	200°

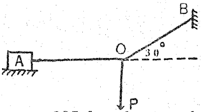
(*b*)					
	Forces in newtons	3000	2150	4300	5000
	Angles	60°	120°	240°	300°

6. If the force $P=100$ kg. acting as shown in the figure and a force Q (not shown) whose line of action is along AB, have a resultant R which passes through the point O, determine the magnitudes of Q and B. The distances OB and OC are given in the figure in centimetres.



7. It is desired to exert a horizon-

tal force of 25 kg. on the body A by means of a vertical pull P as shown in the figure. Find the magnitude of P and the magnitude of the force in the string OB.



8. A telephone pole has five wires radiating horizontally from the top, producing the following horizontal pulls : 200 kg. due south, 190 kg. due east, 205 kg. due north-east, 185 kg. at 30° east of north, 210 kg. due north-west. Find graphically the resultant pull on the top of the pole. Verify the result analytically.

9. $ABCD$ is a quadrilateral, and E the point of intersection of the lines joining the middle points of the opposite sides. O is any point. Prove that the resultant of the forces OA, OB, OC, OD is equal to $4(OE)$. [*Banaras*, 1964]

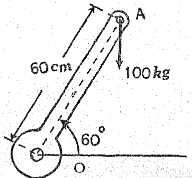
10. Three forces represented by vectors $\overline{PA}$, $\overline{PB}$ and $\overline{PC}$ diverge from a point P and three forces represented by vectors $\overline{AQ}$, $\overline{BQ}$ and $\overline{CQ}$ converge to a point Q. Show that the resultant in magnitude and direction is given by $3\,PQ$ and that it passes through the centroid of the triangle ABC. [*Banaras*, 1940]

11. A 100 kg. vertical force is applied to the end A of a lever attached to a shaft at O. Determine

(i) the moment of the force about O;

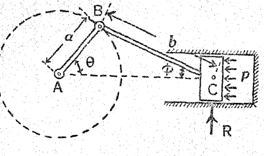
(ii) the magnitude of the horizontal force applied at A which creates the same moment about O;

(iii) the smallest force applied at A which will have the same moment about O.



12. A large heavy block in the shape of a cube, side a, is resting on a horizontal floor; and is to be turned over about an edge by pulling with a crane. Where and in what direction should the cable of the chain be attached if the moment of the pulling force about the edge is to be maximum, the magnitude of the pull remaining the same.

13. Show that if a light string passes round a pulley mounted on smooth bearings, the tensions in the two portions on either side of the pulley are equal.



14. The piston C of the engine shown in the figure has

a radius r, and the gas pressure in the cylinder is p. Determine the thrust in the connecting rod BC and calculate the turning moment M exerted on the crank shaft AB when the angle BAC is θ. Take $AB=a$ and $BC=b$.

[Hint. The force P due to pressure $=\pi r^2 p$. The thrust T along CB is the resultant of the force P, and the reaction R of the cylinder wall acting normal to CA. Therefore, if $\angle BCA=\phi$,

$$T\cos\phi=P=\pi r^2 p, \text{ or } T=\pi r^2 p/\cos\phi. \qquad (1)$$

The turning moment M

$$=T\sin(\theta+\phi)=\pi r^2 p\sin(\theta+\phi)/\cos\phi.$$
$$=\pi r^2 p(\sin\theta+\cos\theta\tan\phi). \quad \ldots \quad (2)$$

Also, from triangle ABC,

$$\sin\theta/b=\sin\phi/a,$$

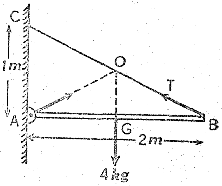
so that $\cos\phi=\sqrt{\{1-(a^2/b^2)\sin^2\theta\}}$.

Substituting for ϕ in (1) and (2), we get T and M.]

2·3. Equilibrium of three forces. We have seen earlier that if a body is in equilibrium under the action of two forces, they must be collinear. We shall now show that if a body is in equilibrium under the action of three coplanar forces, then they must either meet at a point or be parallel.

For, if the forces are not parallel and if two of the forces be combined into a resultant, then in order that the third force balances this resultant, it must act along the same line as the resultant (though in opposite direction). Hence the three forces must meet at a point. This fact is very useful in determining the line of action of an unknown force, if the lines of action of the other two are known.

For example, suppose a rod AB, 2 m. long and of weight 4 kg., is pinned at A to a wall and the other end B is supported by a cord BC, C being 1 m. vertically above A. The rod is kept horizontal by the cord. It is required to obtain the reaction of the pin at A and the tension in the cord BC.



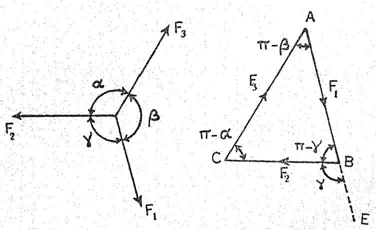
The free-body diagram for the rod gives three forces in equilibrium, the weight of the rod 4 kg.,

acting through its centre of gravity G, the tension T in the cord at B, and the reaction R of the pin at A. These three forces, therefore, must meet at a point. Since the lines of action of the first two forces are known, that of the third force R is obtained by joining A to O, the point of intersection of the first two forces. The magnitudes of tension T and the reaction R can then be found graphically by the triangle of forces or analytically by the methods given below.

2·31. Lami's Theorem. When three concurrent forces acting on a body keep it in equilibrium their magnitudes can be determined from the following theorem known as *Lami's theorem.*

If three concurrent forces in a plane be in equilibrium, then each is proportional to the sine of the angle between the other two.

Let F_1, F_2, and F_3 be three concurrent forces in equilibrium, and let α, β, γ be the angles between the forces.



Represent F_1, F_2 and F_3 by parallel free vectors AB, BC and CD. Since the forces are in equilibrium their resultant vanishes, and so D must coincide with A. Hence F_3 is represented by CA.

Produce AB to E; then since BE and BC are parallel to F_1 and F_2, the angle between them is equal to the angle between F_1 and F_2, namely γ. Hence

$$\angle B=\pi-\angle EBC=\pi-\gamma.$$

Similarly $\angle C=\pi-\alpha$, and $\angle A=\pi-\beta$.

By trigonometry :

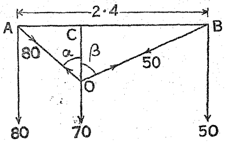
$$\frac{AB}{\sin C}=\frac{BC}{\sin A}=\frac{CA}{\sin B}.$$

$$\therefore \qquad \frac{F_1}{\sin(\pi-\alpha)}=\frac{F_2}{\sin(\pi-\beta)}=\frac{F_3}{\sin(\pi-\gamma)},$$

or

$$\frac{F_1}{\sin\alpha}=\frac{F_2}{\sin\beta}=\frac{F_3}{\sin\gamma}.$$

Ex. 1. A cord, with a weight of 80 kg. fastened at one end and another weight of 50 kg. at the other end, is hung over two smooth pegs which are in the same horizontal line 2·4 m. apart. A weight of 70 kg. is suspended from a knot on this cord between the pegs. Find the vertical distance of the knot below the line of the two pegs when the system is in equilibrium.



Let α and β be the angles made by the cord segments AO and OB with the vertical. The free-body diagram for the knot O gives three forces 70 kg., 80 kg. and 50 kg. in equilibrium.

The resultant of 80 kg. and 50 kg. is balanced by 70 kg. Therefore $70^2=80^2+50^2+2.80.50\cos(\alpha+\beta)$,

or

$$\cos(\alpha+\beta)=\frac{70^2-80^2-50^2}{2\times 80\times 50}=-0·5.$$

Hence, $\alpha+\beta=120°$.

By Lami's theorem: $\dfrac{80}{\sin\beta}=\dfrac{70}{\sin(\alpha+\beta)}=\dfrac{50}{\sin\alpha}.$

$$\therefore \qquad \sin\alpha=\tfrac{5}{7}\sin(\alpha+\beta)=\tfrac{5}{7}\sin 120°=0·6186,$$

$$\sin\beta=\tfrac{8}{7}\sin(\alpha+\beta)=\tfrac{8}{7}\sin 120°=0·9897,$$

so that $\alpha=38°\ 12'$, $\beta=81°\ 48'$.

The vertical distance CO of the knot from line AB is given by

$$AC=CO\tan\alpha,$$

$$BC=CO\tan\beta.$$

$$\therefore \qquad AC+BC=CO(\tan\alpha+\tan\beta),$$

or

$$CO=\frac{AC+BC}{\tan\alpha+\tan\beta}=\frac{2·4}{\tan 38°\ 12'+\tan 81°\ 48'}=0·31 \text{ m.}$$

Ex. 2. A heavy uniform rod of weight W rests with its ends on two smooth inclined planes, whose inclination with the hori-

zontal are α and β ($\alpha<\beta$). If θ be the inclination of the rod with the vertical, prove that

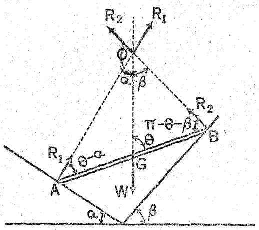
$$\cot\theta=\tfrac{1}{2}(\cot\alpha-\cot\beta),$$

and the reactions of the planes are

$$\frac{W\sin\beta}{\sin(\alpha+\beta)} \text{ and } \frac{W\sin\alpha}{\sin(\alpha+\beta)}$$

respectively. [*Roorkee, Arch.*, 1966]

Suppose the rod AB makes an angle θ with the vertical. The free-body diagram for the rod gives 3 forces, R_1, R_2 and W in equilibrium. Hence, by § 2·3, they must meet at a point O.



From $\triangle OGA$, we have

$$\frac{OG}{GA}=\frac{\sin(\theta-\alpha)}{\sin\alpha}.$$

From $\triangle OGB$, we have

$$\frac{OG}{GB}=\frac{\sin(\theta+\beta)}{\sin\beta}.$$

Since $GA=GB$, therefore

$$\frac{\sin(\theta-\alpha)}{\sin\alpha}=\frac{\sin(\theta+\beta)}{\sin\beta},$$

or $$\frac{\sin\theta\cos\alpha-\cos\theta\sin\alpha}{\sin\alpha}=\frac{\sin\theta\cos\beta+\cos\theta\sin\beta}{\sin\beta},$$

or $$\sin\theta\cot\alpha-\cos\theta=\sin\theta\cot\beta+\cos\theta,$$

or $$2\cos\theta=\sin\theta(\cot\alpha-\cot\beta),$$

or $$\cot\theta=\tfrac{1}{2}(\cot\alpha-\cot\beta).$$

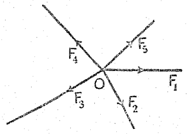
Applying Lami's theorem for the concurrent forces at O,

$$\frac{R_1}{\sin(\pi-\beta)}=\frac{R_2}{\sin(\pi-\alpha)}=\frac{W}{\sin(\alpha+\beta)}.$$

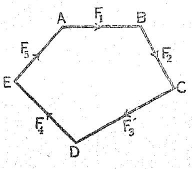
Therefore $$R_1=\frac{W\sin\beta}{\sin(\alpha+\beta)}, \text{ and } R_2=\frac{W\sin\alpha}{\sin(\alpha+\beta)}.$$

2·4. Equilibrium of Concurrent Forces : Graphically. The resultant of coplanar concurrent forces was obtained in § 2·1 as the closing line of a force polygon. If the system is in equilibrium the resultant is zero and the closing line of the force polygon should vanish, that is, the force polygon must close.

Thus if $F_1, F_2, \ldots$ be a system of concurrent forces in equilibrium then the corresponding force diagram



(a) Free-body diagram



(b) Force diagram

$ABCD\ldots$ is a closed force polygon, that is, the final point F coincides with the initial point A (fig. b).

Hence the graphical solution of a problem in equilibrium, consists in plotting in order, to some scale, the forces which are known and then finding the unknown forces by closing the force polygon. Only two unknown quantities can be found out, which may be :

(i) One force unknown in magnitude and direction, or

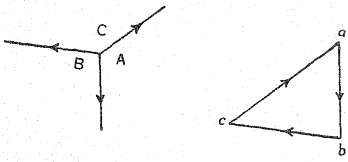
(ii) Two forces unknown in magnitudes but known in directions, or

(iii) Two forces unknown in directions but known in magnitudes.

2·41. Bow's Notation. In graphical solution, it is convenient to use Bow's notation. In this notation the forces in the free-body diagram are denoted by a pair of letters, one on each side of the line of action of the forces and read clockwise around the diagram. The corresponding vectors in the force diagram are denoted with the same lower case letters at each end of the vectors.

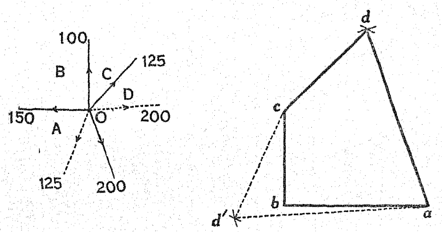
For example, the figure (*a*) below shows three concurrent forces in equilibrium. According to Bow's notation they will be called *AB*, *BC* and *CA*; and the

corresponding force diagram (fig. b) is a triangle of forces abc.



(a) Free-body diagram (b) Force diagram

Ex. 1. A point O is held in equilibrium under the action of 4 forces: one, 150 kg. acting horizontally to the left, another, 100 kg. acting vertically upwards and the third and fourth forces of magnitudes 125 kg. and 200 kg. respectively. Determine the directions of the last two forces.



(a) (b)

Adopting Bow's notation, we have, forces AB=150 kg., BC =100 kg., CD=125 kg., and DA=200 kg. in equilibrium [fig. (a)]. The directions of the last two forces are required. Draw vectors ab=150 kg. and bc=100 kg. (scale 1 cm. =50 kg.) as shown in figure (b). With points c and a as centres draw arcs of radii measuring 125 kg. and 200 kg. respectively to intersect at d and d'. Then $abcd$ is one force polygon and $abcd'$ is another force polygon. The vectors $\overline{cd}$ and $\overline{da}$ (or $\overline{cd'}$ and $\overline{d'a}$) will give the directions of forces 125 kg. and 200 kg. respectively.

By measurement, the angle made with the x-axis by :

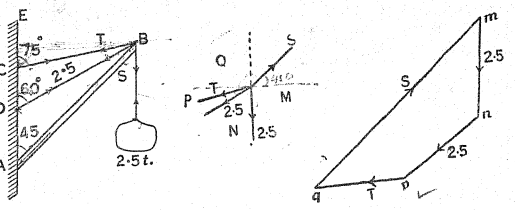
125 kg. force is 46° or 248·5°,

200 kg. force is 290° or 6° .

Ex. 2. A crane consists of a vertical post AE, a jib AB and a tie CB. A cable runs over a smooth pully B and is wound on a drum at D. Angle $BAD=45°$, $BDC=60°$ and $BCE=75°$. Find the forces in CB and AB when a weight of 2·5 tonnes is held by the cable.

Under the action of load 2·5 tonnes, the cable BD is in tension, the force being 2·5 tonnes, as the pulley at B is smooth; the jib AB is in compression and the tie BC is in tension.

Fig. (b) is the free-body diagram for the point B, giving 4 forces in equilibrium: force $MN=2·5$ tonnes, force $NP=2·5$



(a) (b) (c)

tonnes and forces PQ and QM which are required in magnitude, their directions being known. *While naming the forces by Bow's notation, due care is taken to group all the known forces together and the unknown forces together.*

Fig. (c) is the force polygon, drawn to scale 2 cm.$=2·5$ tonnes, with vectors $\overline{mn}$, $\overline{np}$, $\overline{pq}$ and $\overline{qm}$ drawn parallel to forces MN, NP, PQ and QM respectively.

By measurement, vector $\overline{pq}=$ tension in tie $BC=2·2$ tonnes,

vector $\overline{qm}$ $=$compression in jib $AB=6·1$ tonnes.

2·5. Equilibrium of Concurrent Forces: Analytically. I. *Resolved parts of forces.*

The resultant R of concurrent forces is given by

$$R=\sqrt{\{X^2+Y^2\}},$$

where X and Y are the sum of the resolved parts of the

forces along the two axes. For the system to be in equilibrium R must be zero. But R is zero only if

$$X=0 \text{ and } Y=0.$$

Hence the necessary and sufficient conditions for equilibrium are that the sum of the components of all the forces in any two perpendicular directions in the plane of forces be separately zero.

To determine the unknown elements in a problem of equilibrium we write down the equations $X=0$, $Y=0$ and solve them for the unknown elements. Since these are two independent equations, not more than two elements can be found, which may be either two magnitudes or two directions, or one magnitude and one direction.

II. *Moments of forces.*

By the principle of moments, the sum of moments of a system of coplanar concurrent forces about any point is equal to the moment of the resultant about the same point. If the forces are in equilibrium the resultant is zero and so the sum of moments of all the forces about any point A in the plane of forces is zero.

This condition, namely $M_A=0$, must be satisfied for equilibrium. But this is not sufficient as it may only imply that the resultant passes through A. To make this condition sufficient, we take moments about another point B, such that the line joining A and B does not pass through the point of concurrence. Then the equations of equilibrium are:

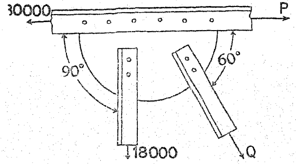
$$M_A=0 \text{ and } M_B=0.$$

An alternative set of equations, obtained by a combination of the above methods, may be expressed as:

$$X=0,\ M_A=0,$$

where A is a point not on the perpendicular from the point of concurrence on the X-axis.

Ex. The figure below represents a panel point of a bridge truss. The stresses in two members are 30,000 kg. and 18,000 kg. as shown. Find the stresses P and Q in the other members.



Resolving the forces along the horizontal and the vertical lines through the joint point,

$P-30000+Q\cos 60°=0.$ (1).

$18000+Q\sin 60°=0.$ . . . (2)

From (2), $Q\sqrt{3}/2=-18000$, or $Q=-20{,}800$ kg.
From (1), $P=30000-Q\cos 60°=30000+20800\times\cdot 5=40{,}400$ kg.

EXAMPLES 3

1. A heavy uniform rod AB, of weight W, is hinged at A and rests inclined at 60° to the horizontal, being acted upon by a horizontal force F applied at the lower end B. Find the reaction at the hinge and the magnitude of the force F.

2. A cyclist, whose weight is 80 kg., puts all his weight upon one pedal of his bicycle when the crank is horizontal and the bicycle is prevented from moving forwards. If the length of the crank is 15 cm. and the radius of the chain-wheel is 10 cm., find the tension of the chain. [*Banaras*, 1965]

3. A body of weight 15 kg. is suspended by two strings, of lengths $2l$ and $3l$, from two points A and B on the same horizontal line. The distance between A and B is $3l$. Find the tensions in the strings.

4. A uniform beam, of length $2a$, rests in equilibrium against a smooth vertical wall and upon a smooth peg at a distance b from the wall. Show that the inclination of the beam to the vertical is

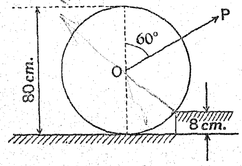
$$\sin^{-1}\left(\frac{b}{a}\right)^{1/3}. \qquad \text{[Roorkee, 1959]}$$

5. A body, whose weight is W, is supported on a certain smooth inclined plane by a force P, acting horizontally. The same body can also be supported by a force Q, acting parallel to the plane. Show that

$$\frac{1}{Q^2}=\frac{1}{P^2}+\frac{1}{W^2}.$$

6. A ring, weight W, can slide on a smooth circular wire of radius r in a vertical plane. The ring is held in a given position

by a string of length l ($l<2r$), attached to the highest point of the vertical circle. Show that the reaction of the wire on the ring is W and the tension in the string is Wl/r.



7. A garden roller has a diameter 80 cm. and weight 100 kg. It is required to pull it over a step 8 cm. high, as shown in the diagram. The handle is pulled in a direction making 60° with the vertical. What force is required just to lift the roller from the ground?

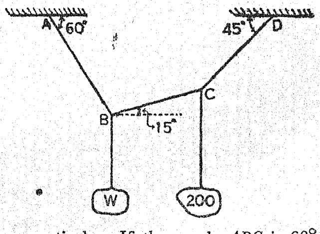
8. A man wishes to pull a lawn roller of diameter 2 feet and weight 200 pounds over a stone 4 inches high. Find the direction in which he should pull so as to raise the roller with the least effort and also the force he applies. [*Banaras*, 1964]

9. ABC is a uniform rod of weight W; it is supported, (B being upper-most) with its end A against a smooth vertical wall AD, by means of a string CD, DB being horizontal and CD inclined to the wall at an angle of 30°. Find the tension in the string and the reaction of the wall and prove that $AC=\frac{1}{3}AB$.

10. A load of $W=2000$ kg. is hung from a pin P at which pieces AP and BP meet like the tie rod and jib of a crane. The angles WPB and WPA are 25° and 60° respectively. Find the forces in AP and BP.

11. The strut of a jib crane is 18 feet long and the tie, 14 feet, is fastened to the upright, 12 feet above the bottom of the strut. If the strut cannot support a thrust greater than one cwt., find the minimum load on the crane which would cause the strut to break. [*Allahabad*, 1963]

12. Two weights of 200 kg. and W kg. are supported by 3 strings, AB, BC and CD, knotted together as shown in the figure. The strings AB, BC and CD make angles of 60°, 15° and 45° with the horizontal respectively. Find the tensions in the strings and the weight W.



13. A light rod AB hangs horizontally by two strings from a point C and weights of 8 kg. and x kg. are attached to A and B respectively. If the angle ABC is 60°

and the tension in the string AC is 10 kg. wt., find the value of x, the tension in the string BC and the angle BAC.

14. A solid cone, of height h and semi-vertical angle α, is placed with its base against a smooth vertical wall and is supported by a string attached to its vertex and to a point on the wall, show that the greatest possible length of the string is

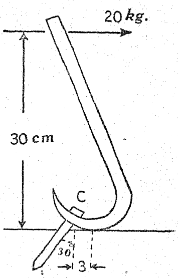
$$h\sqrt{(1+\tfrac{16}{9}\tan^2\alpha)}.$$ [*Roorkee*, 1967]

15. A wheel has five equally spaced radial spokes, all in tension. If the tensions of three consecutive spokes are 2000 kg., 2800 kg. and 2400 kg. respectively, find the tensions in the other two. [*U. P. E. S.*, 1965]

16. A uniform rod rests with one extremity against the inner surface of a smooth fixed hemispherical bowl of radius r; the rim of the bowl is horizontal. If the length of the rod is m times the radius of the bowl ($m \geqslant 2$), show that in the position of equilibrium, the rod makes an angle θ with the horizontal given by

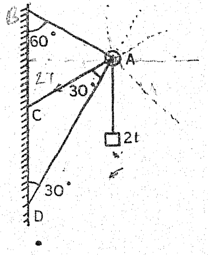
$$4\cos 2\theta = m\cos\theta,$$

and the reaction at the point of the rim of the hemisphere in contact with the rod is $\frac{1}{4}mW$, where W is the weight of the rod.



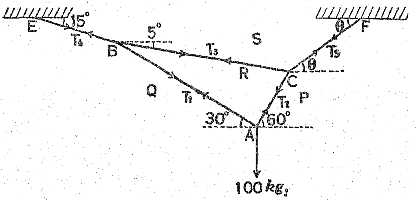
17. A smooth hemispherical bowl of radius 30 cm. is placed so that its edge touches a vertical wall. A uniform rod of weight 5 kg. is placed with one end resting on the inner surface of the bowl and the other end resting against the smooth wall. In the position of equilibrium the rod makes an angle of 30° with the horizontal. Find the reactions of the wall and the bowl and the length of the rod.

18. Find the magnitude of the pull exerted on the nail C in the upper figure if a force of 20 kg. is applied to the handle of the wrecking bar. Also draw the free-body diagram of the bar.



19. A weight of 2 tonnes hangs by a chain passing over a small frictionless pulley A and tied to C. Determine graphically the forces produced in the bars AD and AB, indicating whether they are compressive or tensile. Verify analytically.

20. A weight of 100 kg. is suspended from a system of five cords as shown in the figure. Find the tension in each cord and the angle θ.



[Applying Lami's theorem at A,

$$\frac{100}{\sin 90^\circ}=\frac{T_1}{\sin(90+60)^\circ}=\frac{T_2}{\sin(90+30)^\circ},$$

giving $T_1=100\cos 60^\circ=50$ kg., $T_2=100\cos 30^\circ=86{\cdot}6$ kg.

For the point B

$$\frac{T_1}{\sin(180-10)^\circ}=\frac{T_4}{\sin(30-5)^\circ}=\frac{T_3}{\sin(180-30+15)^\circ},$$

or
$$\frac{50}{\sin 10^\circ}=\frac{T_4}{\sin 25^\circ}=\frac{T_3}{\sin 15^\circ},$$

whence $T_4=121{\cdot}5$ kg., and $T_3=74{\cdot}5$ kg.

Similarly, for the point C,

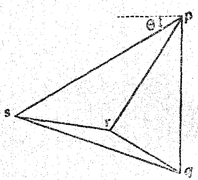
$$\frac{86{\cdot}6}{\sin(\theta+5)}=\frac{74{\cdot}5}{\sin(60-\theta)}=\frac{T_5}{\sin 65^\circ}.$$

$\therefore (86{\cdot}6)(\sin 60^\circ\cos\theta-\cos 60^\circ\sin\theta)$
$=(74{\cdot}5)(\sin\theta\cos 5^\circ+\cos\theta\sin 5^\circ)$

or $\cos\theta\,(75-6{\cdot}5)=\sin\theta\,(43{\cdot}3+74)$,

or $\tan\theta={\cdot}582$. $\therefore$ $\theta=30^\circ\ 12'$.

$\therefore$ $T_5=86{\cdot}6\times\sin 65^\circ/\sin 35^\circ 12'=136$ kg.



Graphical method. The marginal figure is the force diagram for the system. In Bow's notation pqr is the triangle of forces for the forces at A. pq is drawn vertically to represent 100 kg. on a suitable scale. pr and qr are then drawn parallel to T_2 and T_1 respectively. Δqrs can be drawn similarly. Joining p and s gives the triangle prs for the forces at C. Measurement of the sides and θ gives the results.]

CHAPTER III

COPLANAR PARALLEL FORCES

3·1. Resultant of two parallel forces: Let P and Q be two parallel forces acting at the points A and C of a rigid body. Introduce two equal, opposite and collinear forces F at A and C, as shown in the figure. The system is not altered thereby. By the parallelogram law, forces P and F combine to give the resultant R_1; while the forces Q and F combine to give the resultant R_2. Let the two resultants R_1 and R_2 intersect at M. Shift the force R_1 to the point M and resolve it back into the components P and F. Similarly, shift R_2 to M and resolve it into the components Q and F. Then the two forces F at M, being

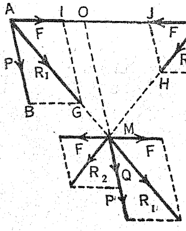


Fig. 1

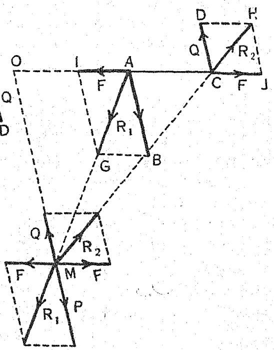


Fig. 2

equal and opposite, cancel each other, while the forces P and Q acting in the same line add up to give a force $P+Q$ (Fig. 1). This is the resultant of the forces P at A and Q at C.

Produce the line of action of $P+Q$ to meet AC at O. Then from the similar triangles AOM and AIG we have

$$\frac{AO}{OM}=\frac{AI}{IG}=\frac{F}{P}. \quad . \quad . \quad . \quad (1)$$

Again, from the similar triangles COM and CJH we have

$$\frac{CO}{OM}=\frac{CJ}{JH}=\frac{F}{Q}. \quad . \quad . \quad . \quad (2)$$

Dividing (2) by (1), we get

$$\frac{CO}{AO}=\frac{P}{Q}. \quad . \quad . \quad . \quad (3)$$

This gives the position of the line of action of the resultant.

Hence *the resultant of two parallel forces of like sense is a parallel force, having a magnitude equal to the sum of the two forces, and whose line of action divides the distance between them internally in a ratio* INVERSELY *proportional to the magnitudes of the two forces.*

If the parallel forces P and Q are unlike in sense ($P > Q$, say), as in fig. 2, the construction for the resultant is still the same. The only difference is that on resolution the forces R_1 and R_2 at M give P and Q in opposite directions. Hence the resultant is $P-Q$ acting at M in the direction of the larger force P. Also, on being produced the line of action of $P-Q$ cuts AC at an *external* point O. The similarity of corresponding triangles still holds, and consequently so does relation (3).

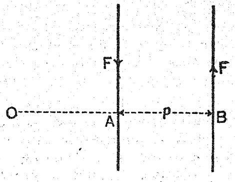
Hence *the resultant of two parallel forces of unlike sense is a parallel force of magnitude equal to the difference of the two forces whose line of action divides the distance between them externally in the ratio inversely proportional to the magnitudes of the two forces.*

If the two forces P and Q are equal but unlike in sense, they cannot be combined into a single resultant,

as in this case the lines of action of R_1 and R_2 will be parallel and will not meet. Such a system of forces is called a couple.

3·2. Couples. Two parallel forces acting on a rigid body, which are equal in magnitude but opposite in sense and have different lines of action, constitute a *couple*.

The perpendicular distance p between the forces is called the *arm of the couple.* The plane in which the two forces lie is called the *plane of the couple*.

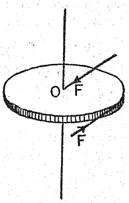


The product of one of the forces with the arm of the couple, i.e. Fp, is called the *moment of the couple.* The sum of the moments of the forces constituting the couple is constant and is same about any point in the plane of the couple. Thus

$$M_0 = -F \times OA + F \times OB = F(OB - OA) = Fp.$$

3·21. Properties of the couple.

(1) When a couple is applied to a body, it causes or tends to cause rotation about an axis, perpendicular to the plane of the couple.



(2) When a force F is applied to a pivoted body away from the axis (see fig.) then the force F and an equal and opposite reaction of the axis form a couple, which causes rotation of the body about the axis.

(3) The turning effect of a couple depends upon its moment and not on the force or the arm of the couple separately. Thus couples of different forces and arms and acting in the same plane (or parallel planes) will have the same turning effect if their moments are the same. Such couples will be equivalent to each other.

(4) It follows that a couple can be rotated through any angle, or be transferred to any other position in its plane (or a parallel plane) without causing a change in the effect it produces.

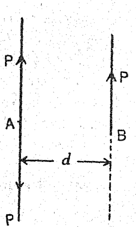
(5) The moment of a couple is a vector quantity whose direction is defined by its axis of rotation. Thus a couple can be represented by a free vector whose magnitude is the moment of the couple and whose direction is the axis of rotation. The composition and resolution of couples can then be performed by the laws of vector addition.

When the axes of several couples are the same they can be algebraically added up. Thus, the moment of the resultant couple of a number of couples in a plane (or in parallel planes) is equal to the algebraic sum of the moments of the various couples.

(6) A couple can be balanced only by another couple in the same (or a parallel) plane, having a moment equal in magnitude but opposite in sense.

(7) A couple C and a force P in the plane of the couple combine to give a force of the same magnitude and direction as P, but the line of action is shifted through a distance d, such that

$$Pd = \text{moment of the couple } C.$$

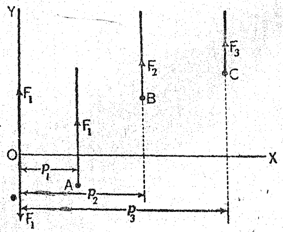


PROOF: Let the force P act at the point A. Since couples of equal moments are equivalent, we may replace the couple C by two equal and opposite forces, $-P$ acting at A and P acting at B, distance d apart, where Pd = moment of the couple C. The force P and $-P$ at A cancel each other, leaving the force P at B at a distance d from the original force.

Notice that the force shifts to its right when the couple has a positive moment.

Conversely a given force P at a point B is equivalent to a force P at any other point A together with a couple C of moment Pd, where d is the perpendicular distance of A from the line of action of the original force.

3·3. Resultant of Coplanar Parallel Forces. Let $F_1, F_2, F_3, \ldots$ be a system of parallel forces, acting on a rigid, body at the points $A, B, C, \ldots$. Take the y-axis parallel to the forces and let $p_1, p_2, p_3, \ldots$ be the distances of these forces from the origin O, respectively.



Introduce equal and opposite forces F_1 and $-F_1$ at O along the y-axis. The system is not altered thereby. The force F_1 at A and $-F_1$ at O form a couple of moment F_1p_1. Hence the force F at A

$\equiv$a force F_1 along y-axis+a couple of moment F_1p_1.

Similarly, the force F_2 at B

$\equiv$a force F_2 along y-axis+a couple of moment F_2p_2,

the force F_3 at C

$\equiv$a force F_3 along y-axis+a couple of moment F_3p_3,

and so on.

Thus the given system is reduced to a set of collinear forces along the y-axis and a set of coplanar couples.

The resultant of the collinear forces is a single force

$$R=F_1+F_2+F_3+\ldots,$$

and the resultant of the coplanar couples is a couple C of moment

$$M_0=F_1p_1+F_2p_2+F_3p_3+\ldots,$$

Four cases arise :

(1) If both R and M_0 are non-zero, the force and the couple combine to give another force R, having its line of action shifted through a distance p such that

$$Rp=M_0, \qquad \ldots \quad (1)$$

or

$$p=\frac{M_0}{R}=\frac{\Sigma F_1p_1}{\Sigma F_1}.$$

(2) If $\Sigma F_1=0$ but $\Sigma F_1p_1\neq 0$, then the resultant of the system is a couple C of moment ΣF_1p_1.

(3) If $\Sigma F_1\neq 0$ but $\Sigma F_1p_1=0$, then the resultant of the system is a single force $R=\Sigma F_1$ along the y-axis.

(4) If both ΣF_1 and ΣF_1p_1 are zero, then the resultant vanishes and the system is in equilibrium.

3·4. Principle of moments for parallel forces. Equation (1) of the last article, namely

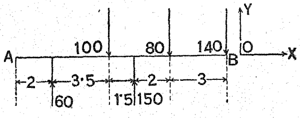
$$Rp=M_0=\Sigma F_1p_1,$$

shows that the moment of the resultant R about O is equal to the sum of the moments of all the forces about O.

Since the choice of the origin O is arbitrary, hence we may state the principle of moments for parallel forces as:

The moment of the resultant of a system of coplanar parallel forces about any point in the plane of the forces is equal to the algebraic sum of the moments of the forces about the same point.

Ex. Determine the magnitude, direction and the distance from A of the resultant of the parallel forces shown in the figure. The magnitudes of forces are given in kilograms and the distances in metres.



The magnitude of the resultant R

$$=60-100+150-80-140=-110.$$

Hence the resultant is 110 kilograms acting downwards. Let x be the distance of the resultant from A. From the principle of moments,

$$110x=-60\times2+100\times5{\cdot}5-150\times7+80\times9+140\times12=1780.$$

$$\therefore \quad x=\frac{1780}{110}=16\tfrac{2}{11}\text{ metres.}$$

3·5. Equilibrium of coplanar parallel forces. It has been shown, in § 3·3, that a system of parallel forces acting at different points may be reduced to a force $R=\Sigma F_1$ through an arbitrarily chosen point O and a couple C of moment $M_0=\Sigma F_1p_1$. Since a force cannot balance a couple, it is necessary for the equilibrium of the forces that the resultant R and the couple C should both vanish.

Hence a set of necessary and sufficient conditions for the equilibrium of parallel forces are:

$$\Sigma F_1=0 \text{ and } \Sigma F_1p_1=0.$$

These equations will determine the unknown elements involved in a problem of equilibrium.

An alternative set of conditions, sufficient to ensure

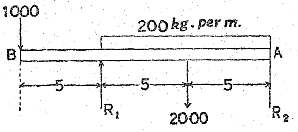
equilibrium is that the sum of moments of the forces about two points in the plane of forces are separately zero, that is

$$\Sigma M_A = 0, \ \Sigma M_B = 0,$$

provided the line joining A and B is not parallel to the forces.

Since there are two independent equations in each set, not more than two unknown elements can be found, which may be the magnitudes of two forces.

Ex. 1. A beam 15 m. long is supported at two points, as shown in the figure. It carries a load of 1000 kg. at the left end and a distributed load of 200 kg. per metre from its right end to a distance of 10 m. Find the reactions of the supports.



Let R_1 and R_2 be the reactions of the supports. The distributed load of 2000 kg. may be taken as a concentrated load acting through the centre of the distributed load, i.e. 5 m. from the right end.

Then $$\Sigma F_1 = R_1 + R_2 - 1000 - 2000 = 0,$$

or $$R_1 + R_2 = 3000.$$

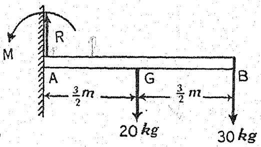
Taking moments about A,

$$-2000 \times 5 + R_1 \times 10 - 1000 \times 15 = 0.$$

Therefore $$R_1 = 2500 \text{ kg.}$$

$$R_2 = 3000 - 2500 = 500 \text{ kg.}$$

Ex. 2. A uniform horizontal beam AB, of length 3 metres and weight 20 kg., is fixed rigidly to a wall at its end A; and a weight of 30 kg. is suspended from the other end B which is free (i.e. unsupported). Find the reaction exerted by the wall on the beam.



The weight of the beam, which will act at its centre G, and the weight 30 kg. will give a resultant acting somewhere between G and B.

This resultant cannot be balanced by a single force acting at A. Hence the reaction of the wall must consist of a vertical force R together with a couple of moment M.

Resolving the forces vertically, we get

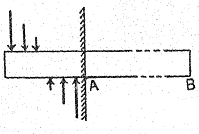
$$R=20+30=50 \text{ kg.}$$

Taking moments about A, we get

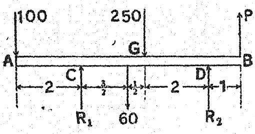
$$M=20\times\tfrac{3}{2}+30\times3=120 \text{ kg.m.}$$

Hence the reaction of the wall at A consists of a force of 50 kg. acting vertically upwards and a couple of positive moment 120 kg-m.

NOTE. A beam rigidly fixed at one end and free at the other is called a *cantilever*. The fixed end is generally built into the wall. The distributed reactions acting on the built in portion give rise to the resultant R and M.



Ex. 3. A uniform beam, length 7 metres and weight 60 kg., rests in a horizontal position on two supports which are 4 metres apart. It overhangs the right support by 1 metre and carries a load of 250 kg. mid-way between the supports and 100 kg. at the left end. What upward force at the right end would be necessary to just tilt the beam? [*Roorkee*, 1965]

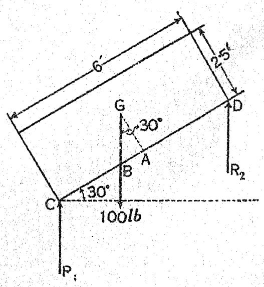


Let P be the value of the force applied upwards at the right end when it starts to tilt about the support C. Then the beam is just lifted from the support at D and so the reaction at D reduces to zero. Hence taking moments about C in the limiting case

$$\Sigma M_c=P\times5-250\times2-60\times\tfrac{3}{2}+100\times2=0,$$

or $\quad 5P=500+90-200. \quad \therefore \quad P=78$ kg.

Ex. 4. A box, 6 ft. long and cross-section 2·5 ft. square, is carried up a stair case by two men who hold it by the front and back edges of its lower face. This face is inclined at 30° to the horizontal when it is being carried up. The box weighs 100 lb. and its weight acts through the geometric centre of the box. How much weight does each man support? [*Allahabad*, 1965]



The forces acting on the box are shown in the figure, R_1 and R_2 being the vertical forces applied by the two men.

Equating forces:

$$R_1+R_2=100.$$

Taking moments about C: $R_2 \times CD \cos 30° = 100 \times CB \cos 30°$,

or $$R_2 \times CD = 100 \times CB.$$

Now $$CB = CA - BA = 3 - GA \tan 30°$$
$$= 3 - \tfrac{5}{4}(1/\sqrt{3}) = 2{\cdot}28 \text{ ft.}$$

$\therefore$ $$R_2 = 100 \times CB/CD = 100 \times 2{\cdot}28/6 = 38 \text{ lb.}$$
$$R_1 = 100 - R_2 = 62 \text{ lb.}$$

Examples 4

1. The resultant of two unlike parallel forces is 2 kg. and acts at distances of 60 cm. and 80 cm. from them. Find the forces.

2. Two like parallel forces P and Q act on a rigid body at A and B respectively. If P and Q be interchanged, show that the point of application of the resultant is displaced through a distance d along the line AB given by

$$d = \frac{P-Q}{P+Q} AB.$$

3. Find the magnitude and the position of the resultant of the parallel force system given below :—

(a)	Force F kg. :	$+15$	-30	$+25$	$+10$	
	distance x m. :	-1	1	2	3	
(b)	Force F kg. :	$+100$	-50	$+80$	-30	
	distance x m. :	0	2	4	6	
(c)	Force F kg. :	$+40$	$+50$	-20	$+10$	$+60$
	distance x m. :	-4	-2	0	2	4

4. Three like parallel forces P, Q and R act at the points A, B and C of a triangle ABC. If these forces P, Q and R are in the ratio $\sin 2A : \sin 2B : \sin 2C$ respectively, prove that the resultant will pass through the circumcentre of the triangle.

5. A heavy uniform rod, 2 metres long, rests horizontally on two pegs, 50 cm. apart. A weight of 5 kg. suspended at one end or 2 kg. suspended at the other end will just tilt the rod up. Find the weight of the rod and the distances of the pegs from the centre of the rod.

6. Two men, one stronger than the other, have to remove a slab weighing 300 kg. with a light pole of length 6 m. The weaker man cannot carry more than 100 kg. Where must the

stone be fastened to the pole so that the weaker man has just his full share of the load? [*Ranchi*, 1965]

7. A uniform beam, length 10 metres and weight 400 kg., rests in a horizontal position on two supports, which are 7·5 m. apart. It overhangs the left support by 1 metre. It carries a load of 600 kg. at the left end and 2000 kg. mid-way between the supports. What upward vertical force at the right end would be necessary to tilt the beam?

8. A beam AB, 10 feet long and weighing 6 lb., is supported at A and at another point. A load of 1 lb. is suspended at B, and loads of 5 lb. and 4 lb. at points 3 feet and 6 feet from B. If the pressure on the support A is 4 lb. wt., where is the other support? [*Banaras*, 1954]

9. A uniform rod, length l and weight W, is supported horizontally at the two ends; and n equal weights, w each, are placed at equal distances a, starting from the right end of the rod $[(n-1)a<l]$. Find the reactions at the supports.

10. A uniform beam, 8 metres long, is simply supported at the two ends. Two concentrated loads of 500 kg. and 250 kg. are applied at 2 m. and 6 m. from the left end. Calculate the length of the beam from the right end over which a uniformly distributed load of 75 kg. per metre can be applied so that the reactions at the two ends may become equal.

11. A beam, 6 metres long and weighing 25 kg. per metre, rests horizontally on two supports 4 metres apart and orverhangs the left support by 80 cm. It carries a distributed load of 100 kg. per metre run between the supports and two concentrated loads of 200 kg. and 300 kg. at 1·2 m. from left end and 1·6 m. from right end respectively. Find the reactions at the supports.

12. A beam, weighing 10 kg. per metre, rests horizontally on two supports; one is 1 metre from the left end and the other is 1·5 metres from the right end. It carries a load of 30 kg. at 4 metres from the left end. If the reaction at the right support is twice that of the left support, find the length of the beam.

13. Three spur wheels, whose shafts pass through their centres A, B, C have 42, 30, 16 teeth respectively, and are geared with B between A and C. Two counter clockwise couples 49 and 25 ft.-lb. are applied to the wheels A and B respectively. Find the couple that must be applied to the shaft of C, so that equilibrium may be maintained. [*Roorkee*, 1966]

14. There are three bricks, each 9 inches long, of same shape and material. The lowest one lies on the ground and the other two are placed upon it so that each projects x inches over the one immediately beneath it. If the bricks are just on the point of falling over, find the value of x. [*Allahabad*, 1963]

15. A brick is placed with its flat side on the ground and three bricks are placed similarly on its top. The top brick is pulled forward as far as possible in the direction of its length; then the first two together over the third and so on. Show that when all the bricks have been pushed forward in this way as far as possible the top brick overhangs the one at the bottom by a distance

$$\tfrac{1}{2}L(1+\tfrac{1}{2}+\tfrac{1}{3}),$$

where L is the length of the brick. [*U.P.E.S.*, 1964]

16. A string of length 1·5 m. is attached to two ends of a rod, 1 m. long and of negligible weight. The string is hung over a smooth peg and two bodies of weights 2 kg. and 3 kg. are attached to the ends of the rod. Find the angle between the two segments of the string at the peg when there is equilibrium, and the tension in the string.

17. A uniform rod AB of length l and weight W is hinged at A and hangs freely from the hinge. A couple of moment G is applied to it in a vertical plane. Find the inclination of the rod to the vertical when it is in equilibrium. [*Roorkee*, 1961]

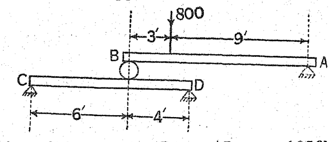
18. Two uniform rods AB, BC, rigidly jointed at B so that ABC is a right angle, hang freely in equilibrium from a fixed point at A. The lengths of the rods are a and b and their weights are wa and wb. If θ be the inclination of AB to the vertical, prove that

$$\tan\theta = b^2/(a^2+2ab).$$

[*Roorkee*, 1964]

19. Two uniform rods AB and BC, of lengths $2l$ and $3l$, and weights $2w$ and $3w$ respectively, are smoothly hinged at B and rest in a horizontal position on two supports, one being at A and the other at a point P between B and C. Find the distance of P from B and the pressures on the supports.

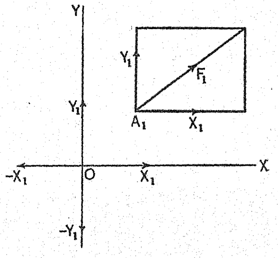
20. The left end of a beam AB is supported on a smooth roller B, resting on another beam CD as shown. The distances between the supports are : AB $=12$ ft., $BC=6$ ft., BD $=4$ ft. If a load of 300 lb. is applied to the beam AB at a point 3 ft. from B, find the reaction of the support D. [*Banaras*, 1956]



COPLANAR NON-CONCURRENT FORCES

4·1. Resultant of coplanar non-concurrent forces. Let F_1, F_2, F_3, ... be a system of coplanar forces acting on a rigid body at $A_1(x_1, y_1)$, $A_2(x_2, y_2)$, $A_3(x_3, y_3)$, ... with reference to a set of rectangular axes OX and OY. Let X_1, Y_1; X_2, Y_2; X_3, Y_3; ... be the components of the forces F_1, F_2, F_3, ... respectively, parallel to the axes.

Introduce equal and opposite forces X_1 and $-X_1$ along OX, and Y_1 and $-Y_1$ along OY at the origin. The system is not altered thereby. The forces Y_1 at A_1 and $-Y_1$ at O form a couple of moment x_1Y_1; while the forces X_1 at A_1 and $-X_1$ at O form a couple of moment $-y_1X_1$. Hence the force F_1 at $(x_1 y_1)$ $\equiv$ Forces X_1 and Y_1 at the origin $+$ a couple of moment $(x_1Y_1 - y_1X_1)$.



Similarly, the forces:

F_2 at $(x_2, y_2) \equiv X_2$, Y_2 at the origin $+$ a couple of moment $(x_2Y_2 - y_2X_2)$,

F_3 at $(x_3, y_3) \equiv X_3$, Y_3 at the origin $+$ a couple of moment $(x_3Y_3 - y_3X_3)$, etc.

Thus the given system is replaced by the forces X_1, X_2, X_3, ... along OX, Y_1, Y_2, Y_3, ... along OY, and a set of coplanar couples of moments $(x_1Y_1 - y_1X_1)$, $(x_2Y_2 - y_2X_2)$, $(x_3Y_3 - y_3X_3)$, ...

The resultant of the forces at O is a force R given by:

$$R = \sqrt{\{(\Sigma X_1)^2 + (\Sigma Y_1)^2\}}$$

making an angle θ with the x-axis where

$$\theta = \tan^{-1}\{(\Sigma Y_1)/(\Sigma X_1)\}.$$

The resultant of the coplanar couples is a couple C of moment $\Sigma(x_1 Y_1 - y_1 X_1)$.

The given system of forces is thus reduced to a force R acting at O, together with a couple C. Four cases arise:

(1) If both R and C are non-zero, they combine to give the resultant of the given system (§ 3·21), which is of the same magnitude and direction as R but the line of action is shifted through a distance p such that

$$Rp = \Sigma(x_1 Y_1 - y_1 X_1). \tag{1}$$

The components of R parallel to the axes are ΣX_1 and ΣY_1. If $P(x, y)$ be any point on the line of action of the final resultant R, then by the principle of moments (§ 2·21).

$$Rp = x\Sigma Y_1 - y\Sigma X_1.$$

Hence the resultant of the system is a single force R whose line of action has the equation

$$x\Sigma Y_1 - y\Sigma X_1 = \Sigma(x_1 Y_1 - y_1 X_1).$$

(2) If $R=0$ but $\Sigma(x_1 Y_1 - y_1 X_1)$ is not zero, then the resultant of the system is a couple C of moment $\Sigma(x_1 Y_1 - y_1 X_1)$.

(3) If $\Sigma(x_1 Y_1 - y_1 X_1) = 0$ but R is not zero, then the resultant of the system is a single force R through the origin O.

(4) If $R=0$ and $\Sigma(x_1 Y_1 - y_1 X_1) = 0$ then the system is in equilibrium.

4·2. The Principle of Moments for Coplanar Forces. From equation (1) of the last article, we have

$$\begin{aligned} Rp &= \Sigma(x_1 Y_1 - y_1 X_1) \\ &= \Sigma F_1 p_1, \text{ by § 2·2.} \end{aligned}$$

This shows that the moment of the resultant R about the point O is equal to the sum of the moments of forces F_1, F_2, F_3, … about O.

Since the choice of the origin is arbitrary, we may state the principle of moments for coplanar forces as follows:

The moment of the resultant of a system of coplanar forces about any point in the plane of the forces is equal to the algebraic sum of the moments of the forces of the system about that point.

Ex. 1. If a system of forces in one plane reduces to a couple whose moment is H, and when each force is turned round its point of application through a right angle it reduces to a couple of moment V; prove that when each force is turned through an angle α, the system is equivalent to a couple whose moment is

$$H \cos \alpha + V \sin \alpha.$$

Let the forces F_1, F_2, ... of the system act respectively at the points (x_1, y_1), (x_2, y_2), ... at angles θ_1, θ_2, ... to the x-axis. Since $R = 0$, therefore

$$\Sigma X_1 = \Sigma F_1 \cos \theta_1 = 0,$$
$$\Sigma Y_1 = \Sigma F_1 \sin \theta_1 = 0,$$

and the moment of couple $H = \Sigma(x_1 Y_1 - y_1 X_1)$

$$= \Sigma(x_1 F_1 \sin\theta_1 - y_1 F_1 \cos \theta_1)$$
$$= \Sigma\{F_1(x_1 \sin \theta_1 - y_1 \cos \theta_1)\}. \qquad (1)$$

When the forces are turned through 90°, we get

$$\Sigma X_1 = \Sigma F_1 \cos(\theta_1 + 90°) = \Sigma(-F_1 \sin \theta_1) = 0,$$
$$\Sigma Y_1 = \Sigma F_1 \sin(\theta_1 + 90°) = \Sigma F_1 \cos \theta_1 = 0,$$

and

$$V = \Sigma\{x_1 F_1 \sin(\theta_1 + 90°) - y_1 F_1 \cos(\theta_1 + 90°)\}$$
$$= \Sigma\{F_1(x_1 \cos \theta_1 + y_1 \sin \theta_1)\}. \quad . \quad . \quad (2)$$

When the forces are turned through an angle α, then

$$\Sigma X_1 = \Sigma F_1 \cos(\theta_1 + \alpha) = \Sigma(F_1 \cos \theta_1 \cos \alpha - F_1 \sin \theta_1 \sin \alpha)$$
$$= \cos \alpha \, \Sigma F_1 \cos \theta_1 - \sin \alpha \, \Sigma F_1 \sin \theta_1 = 0,$$
$$\Sigma Y_1 = \Sigma F_1 \sin(\theta_1 + \alpha) = \Sigma(F_1 \sin \theta_1 \cos \alpha + F_1 \cos \theta_1 \sin \alpha)$$
$$= \cos \alpha \, \Sigma F_1 \sin \theta_1 + \sin \alpha \, \Sigma F_1 \cos \theta_1 = 0,$$

and the moment of the couple

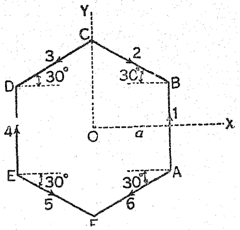
$$= \Sigma\{x_1 F_1 \sin(\theta_1 + \alpha) - y_1 F_1 \cos(\theta_1 + \alpha)\}$$
$$= \Sigma\{x_1 F_1(\sin \theta_1 \cos \alpha + \cos \theta_1 \sin \alpha) - y_1 F_1(\cos \theta_1 \cos \alpha - \sin \theta_1 \sin \alpha)\}$$

$$=\cos\alpha\,\Sigma\{F_1(x_1\sin\theta_1-y_1\cos\theta_1)\}+\sin\alpha\,\Sigma\{F_1(x_1\cos\theta_1+y_1\sin\theta_1)\}$$

$$=H\cos\alpha+V\sin\alpha,\text{ from (1) and (2).}$$

Ex. 2. *ABCDEF* is a regular hexagon and *O* is its centre. Forces of magnitude 1, 2, 3, 4, 5 and 6 act in the line *AB*, *CB*, *CD*, *ED*, *EF*, *AF* in senses indicated by the order of the letters. Find the resultant of the system and find the point in *AB* through which the resultant passes. [*I. C. S.*, 1938]

Forces are acting as shown in the figure. It is easy to see that the six forces are acting at the angles 90°, −30°, 210°, 90°, −30°, 210° with x-axis. Hence the sum of their resolved parts along the axes are :



$$\Sigma X_1=1\cos 90°+2\cos(-30°)+3\cos 210°+4\cos 90°+5\cos(-30°)+6\cos 210°$$
$$=-\sqrt{3}.$$
$$\Sigma Y_1=1\sin 90°+2\sin(-30°)+3\sin 210°+4\sin 90°+5\sin(-30°)+6\sin 210°$$
$$=-3.$$

Moment of the forces about O

$$=1.a-2.a+3a-4a+5a-6a=-3a.$$

Therefore, the magnitude of the resultant

$$R=\sqrt{\{(\Sigma X_1)^2+(\Sigma Y_1)^2\}}=\sqrt{(3+9)}=2\sqrt{3},$$

and for its direction $\tan\theta=\Sigma Y_1/\Sigma X_1=-3/(-\sqrt{3})=\sqrt{3}$, in the third quadrant. Hence

$$\theta=240°.$$

The equation of line of action of R is given by :

$$x\Sigma Y_1-y\,\Sigma X_1=\Sigma(x_1Y_1-y_1X_1),$$

i.e.,
$$-3x+y\sqrt{3}=-3a.$$

In order to find the interesection with line AB, put $x=a$ in the above equation. We find that $y=0$. Hence the resultant intersects AB atits middle point.

Ex. 3. A body is acted by a system of coplanar forces, (16 kg., 0°) at (0, 6), (25 kg., 30°) at (8, 6), (30 kg., 225°) at (8, 0) and (40 kg., 120°) at (−6, −8). Find the resultant and its equation.

It would be convenient to tabulate the solution as below :—

Forces with direction	Coordinates	X	Y	$M_0=xY-yX$
16 kg., 0°	(0, 6)	16	0	$0-96 \quad =-96$
25 kg., 30°	(8, 6)	21·6	12·5	$100-129{\cdot}6 \quad =-29{\cdot}6$
30 kg., 225°	(8, 0)	−21·2	−21·2	$-169{\cdot}6-0 \quad =-169{\cdot}6$
40 kg., 120°	(−6, −8)	−20	34·64	$-207{\cdot}84-160=-367{\cdot}84$
		−3·6	25·94	$=-663{\cdot}04$

$\therefore \qquad R=\sqrt{\{(\Sigma X_1)^2+(\Sigma Y_1)^2\}}=\sqrt{\{(3{\cdot}6)^2+(25{\cdot}94)^2\}}=26{\cdot}2$ kg.

$$\tan\theta=\frac{\Sigma Y_1}{\Sigma X_1}=\frac{25{\cdot}94}{-3{\cdot}6}=-7{\cdot}206. \qquad \therefore \quad \theta=97°\ 54'.$$

Equation of the line of action of R is given by $x\Sigma Y-y\,\Sigma X=M_0$,

or $$25{\cdot}94\,x+3{\cdot}6\,y=-663{\cdot}04,$$

or $$7{\cdot}2\,x+y+184{\cdot}5=0.$$

Examples 5

1. A rigid body is acted on by the following forces:

(30 kg., 45°) at (2, 1), (40 kg., 135°) at (0, 0), (10 kg., 270°) at (2, −1), and (20 kg., 0°) at (−1, 3).

(*a*) Replace the system by a single force through the origin and a couple; (*b*) find the resultant and the equation of its line of action.

2. *ABCD* is a square of 2 inch side, *BD* being the diagonal. A force of 50 lb. acts along *BC* from *B* towards *C*; a force of 80 lb. acts along *CD* from *C* towards *D*; and a force of 60 lb. acts along *DB* from *D* towards *B*. Replace these forces by two equivalent forces one of which acts at *A* along the line *AD*. [*Banaras*, 1956]

3. Three forces, acting upon a rigid body, are represented in magnitude, direction and the line of action by the sides of a triangle taken in order. Prove that they are equivalent to a couple whose moment is represented by twice the area of the triangle. [*Roorkee*, 1965]

4. *ABC* is an equilateral triangle. Forces 4 kg., 2 kg., and 1 kg., act along the sides *AB*, *AC* and *BC*. Find their resultant in magnitude and direction. Find also the point at which its line of action meets *BC*.

5. Forces equal to $3P$, $7P$ and $5P$ act along the sides *AB*, *BC* and *CA* of an equilateral triangle *ABC*. Find the magnitude and the direction of the resultant.

6. *ABC* is an isosceles triangle whose angle *A* is 120° and forces of magnitudes 1, 2 and 4 kg. wt. act along *AB*, *AC* and *BC* respectively; find the magnitude and the direction of the resultant and also find the point in which its line of action meets the side *BC*.

7. Three forces *P*, *Q* and *R* act along the sides *BC*, *CA* and *AB* respectively of a triangle *ABC*, taken in order. Show that if the resultant passes through

(*a*) the centroid of the triangle, then

$$\frac{P}{\sin A}+\frac{Q}{\sin B}+\frac{R}{\sin C}=0;$$ [*Roorkee*, 1962]

(*b*) the circumcentre of the triangle, then

$$P\cos A+Q\cos B+R\cos C=0;$$ [*Banaras*, 1945]

(*c*) the orthocentre of the triangle, then

$$P\sec A+Q\sec B+R\sec C=0.$$ [*Roorkee*, 1960]

8. Three forces *P*, *Q* and *R* act along the sides of the triangle formed by the lines :

$$x+y=1,\ x-y=1,\ y=0,$$

in the same order (anticlockwise). Find the magnitude and the equation of the line of action of their resultant.

9. Forces P, $2P$, $3P$ and $4P$ act along the sides *AB*, *BC*, *CD* and *DA* respectively of a square *ABCD*. Find the magnitude of their resultant and the points in which it cuts the sides *AB* and *BC*.

10. *ABCD* is a rectangle whose side *AB* is 4 ft. and *AD* is 3 ft. Forces 40 lb., 30 lb., 60 lb. and 70 lb. act along the sides *AB*, *CB*, *DC* and *DA* respectively. Taking *AB* and *AD* as the axes of the coordinates, show that the equation of the line of action of the resultant is

$$x+y=3.$$

11. Forces 1, 2, 4 and 5 newtons act in the clockwise direction along the sides of a square taken in order. Prove that the resultant is parallel to a diagonal, and find where it cuts the side along which the first force acts.

12. The lines of action of the four coplanar forces, (4 kg., 30°), (8 kg., 150°), (5 kg., 240°) and (7 kg., 300°), cut the x-axis at

-2 m., 1 m., 4 m., and 6 m. respectively from the origin. Find their resultant and the point where it cuts the x-axis.

4·3. Equilibrium of coplanar non-concurrent forces. It has been shown in § 4·1 that a system of coplanar forces acting at different points may be reduced to a force R through an arbitrary point O and a couple C. Since a force cannot balance the couple C, it is necessary for the equilibrium of the system that the resultant force R and the couple C should both vanish. Since

$$R=\sqrt{\{(\Sigma X_1)^2+(\Sigma Y_1)^2\}},$$

therefore R vanishes if ΣX_1 and ΣY_1 are separately zero; and C vanishes if the moment M_0 of all the forces about any point O is zero.

Hence a necessary and sufficient set of conditions for the equilibrium of coplanar forces, is

$$\left.\begin{array}{r}\Sigma X_1=0,\\ \Sigma Y_1=0,\\ M_0=0.\end{array}\right\} \quad \ldots \quad (1)$$

These are, therefore, the equations for equilibrium, which will determine the unknown elements involved in a problem of equilibrium.

Other sets of conditions which are necessary as well as sufficient for ensuring equilibrium of a system of coplanar forces are

$$\left.\begin{array}{r}\Sigma X_1=0,\\ M_A=0,\\ M_B=0,\end{array}\right\} \quad \ldots \quad (2)$$

where the line joining the points A and B is not perpendicular to x-axis; or

$$\left.\begin{array}{r}M_A=0,\\ M_B=0,\\ M_C=0,\end{array}\right\} \quad \ldots \quad (3)$$

where the points A, B and C are not collinear. The sets (2) and (3) are equivalent to (1), and may be taken as alternative equations of equilibrium for the system of coplanar forces. The proof that they are necessary and sufficient is left as an exercise for the student.

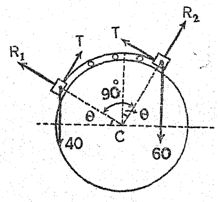
Since there are three independent equations in each set, hence not more than three unknown elements can be determined, which may be

(*a*) the magnitudes of three forces, directions being known, or

(*b*) the magnitude and direction of one force and the magnitude of another, its direction being known.

In applying the equations of equilibrium, the axes should be selected in such a manner so as to simplify the solution. A free-body diagram should always be drawn, before writing down the equations of equilibrium.

Ex. 1. Two bodies of weights 40 kg. and 60 kg. rest on a smooth cylinder of radius a and are connected by means of a cord. The cord is of such a length that it subtends an angle of 90° at the centre of the circular section of the cylinder (see figure). Find the reactions of the cylinder on the bodies, the tension in the cord and the value of θ.



Let R_1, R_2 be the reactions on 40 kg. and 60 kg. bodies respectively, and let T be the tension in the cord. Taking moments about the centre C of the cylinder, we have

$$40 \,.\, a \cos \theta = Ta,$$

for the first body; and

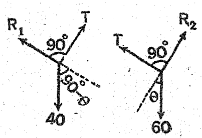
$$60 \,.\, a \sin \theta = Ta,$$

for the second body.

By division, $\frac{3}{2} \tan \theta = 1$,

or $\qquad \theta = \tan^{-1} \frac{2}{3} = 33° .42'$.

Referring to the free-body diagrams for the 40 kg. and 60 kg. bodies and applying Lami's theorem, we have



$$\frac{40}{\sin 90°} = \frac{T}{\sin(180-90+\theta)}$$

$$= \frac{R_1}{\sin(90+90-\theta)},$$

or $\qquad 40 = \dfrac{T}{\cos \theta} = \dfrac{R_1}{\sin \theta},$

$\therefore \quad T = 40 \cos \theta = 40 \times \cdot 832 = 33 \cdot 3$ kg.

and $\quad R_1 = 40 \sin \theta = 40 \times \cdot 555 = 22 \cdot 2$ kg.

$$\frac{60}{\sin 90^\circ}=\frac{T}{\sin(180-\theta)}=\frac{R_2}{\sin(90+\theta)},$$

or $$R_2=60\cos\theta=60\times\cdot 832=49\cdot 9 \text{ kg.}$$

Ex. 2. A wall bracket (fig. (1) below) consists of a horizontal bar AB attached to the wall at A by means of a smooth pin and supported by a cord CB, which is attached to the bar at B and to the wall at C. Find the tension T in the cord and the pin-reaction R at A. The weight of the bar is 200 kg. and it carries two loads of 400 kg. and 1000 kg., as shown.

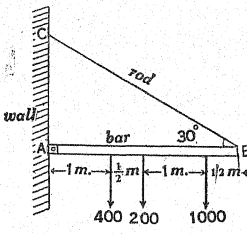


Fig. 1

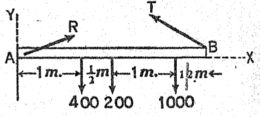


Fig. 2

Free-body diagram for bar AB is drawn in fig. (2). Let R make an angle θ with AB. Three elements are unknown, namely T, R and θ. The equations of equilibrium are

$$\Sigma X_1=R\cos\theta-T\cos 30^\circ=0. \quad \ldots \quad (1)$$

$$\Sigma Y_1=R\sin\theta+T\sin 30^\circ-1600=0. \quad \ldots \quad (2)$$

$$\Sigma M_A=T\sin 30^\circ\times 3-400\times 1-200\times\tfrac{3}{2}-1000\times\tfrac{5}{2}=0. \quad (3)$$

From (3) $\quad \frac{3}{2}T=3200$, or $T=2133\cdot 3$ kg.

From (2) and (1) $\quad R\sin\theta=533\cdot 3,$

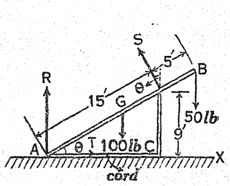
and $\quad R\cos\theta=1847.$

$$\therefore \quad R=\sqrt{\{(533\cdot 3)^2+(1847)^2\}}=1920 \text{ kg.}$$

$$\tan\theta=\frac{533\cdot 3}{1847}=0\cdot 289, \text{ or } \theta=16^\circ\,6'.$$

Ex. 3. A bar AB, weighing 100 lb., leans against a smooth vertical post and rests with its lower end on a smooth horizontal plane (see figure). It is prevented from slipping by the cord AC, attached to A as shown. A weight of 50 lb. is hung from B. Find the reactions of the plane at A and of post at B, and also find the tension in the cord AC.

Let the reaction at A, normal to the plane, be R; the reaction of the post, which is perpendicular to the bar be S; and the tension in the cord be T.



The free-body diagram for the bar AB gives five forces in equilibrium, out of which R, S and T are unknown, their directions being known.

The equations of equilibrium are:

$$\Sigma X_1 = T - S\sin\theta = 0. \quad \ldots \quad (1)$$

$$\Sigma Y_1 = R + S\cos\theta - 150 = 0. \quad \ldots \quad (2)$$

$$\Sigma M_A = S\times 15 - 100\times 10\cos\theta - 50\times 20\cos\theta = 0. \quad (3)$$

From the given dimensions, we find that

$$\sin\theta = \tfrac{9}{15} = \tfrac{3}{5}, \text{ so that } \cos\theta = \tfrac{4}{5}.$$

Therefore from (3):

$$15\,S = 2000\cos\theta = 2000\times\tfrac{4}{5} = 1600,$$

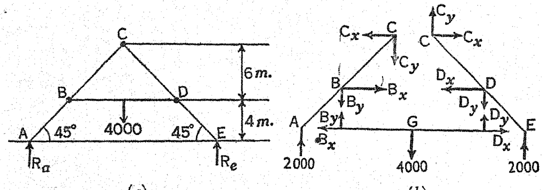
or $$S = 1600/15 = 106{\cdot}7 \text{ lb.}$$

From (1): $$T = S\sin\theta = \frac{1600}{15}\times\frac{3}{5} = 64 \text{ lb.}$$

From (2): $$R = 150 - S\cos\theta = 64{\cdot}7 \text{ lb.}$$

Ex. 4. The A-frame, shown below, carries a load of 4000 kg. at the mid-point of the member BD. It rests on a smooth floor at A and E. Find the reaction of the pin at B on the member ABC, of the pin at D on the member CDE and of the pin at C on the member ABC.

Since the frame and the loading is symmetrical, hence the reactions at A and E are equal. Therefore, considering the equilibrium of the whole frame, $R_a = R_e = 2000$ kg.



(a) (b)

Fig. (b) gives the free-body diagrams for the various members. From the free-body diagram for the member BD, we have

$$\Sigma Y_1 = B_y + D_y - 4000 = 0,$$

and since $BG = GD$, therefore $B_y = D_y = 2000$ kg.

$$\Sigma X_1 = D_x - B_x = 0. \quad \therefore \quad D_x = B_x.$$

From the free-body diagram for the member ABC

$$\Sigma X_1 = B_x - C_x = 0. \quad \therefore \quad B_x = C_x.$$

$$\Sigma Y_1 = 2000 - B_y - C_y = 0. \therefore \quad B_y + C_y = 2000. \quad . \quad . \quad (1)$$

$$\Sigma M_c = B_x \times 6 + B_y \times 6 - 2000 \times 10 = 0.,$$

or $$B_x = \frac{20000}{6} - 2000 = 1333{\cdot}3 \text{ kg.}$$

since $B_y = 2000$ kg.

Hence the components of reaction at B and D are

$$B_x = D_x = 1333{\cdot}3 \text{ kg},$$

$$B_y = D_y = 2000 \text{ kg}.$$

The components of reaction at C are

$$C_x = B_x = 1333{\cdot}3 \text{ kg.},$$

and from (1), $$C_y = 2000 - B_y = 0.$$

Therefore the reaction at B and D

$$= \sqrt{(B_x^2 + B_y^2)} = \sqrt{\{(1333{\cdot}3)^2 + (2000)^2\}} = 2400 \text{ kg}.$$

and $\tan \theta = B_y / B_x = 1333{\cdot}3/2000.$

$$\therefore \quad \theta = \tan^{-1}(2/3) = 33^\circ 40'.$$

The reaction at $C = \sqrt{(C_x^2 + C_y^2)} = C_x = 1333{\cdot}3$ kg.

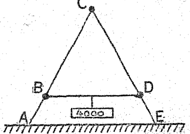
EXAMPLES 6

1. A bar AB, 4 m. long and weighing 25 kg., is hinged at A on a level ground and is resting on a vertical post 1·5 m. high, at a distance of 2 m. from A. A load of 100 kg. is suspended from the end B of the bar. Find the reaction of the post and the reaction of the hinge at A.

2. A ladder rests at an angle of 60° to the horizontal on a smooth floor and against a smooth wall, the lower end being joined by a string to the junction of the wall and the floor. If the ladder weighs 60 kg., find the tension in the string. Find also the tension of the string when a man, weighing 80 kg., has ascended three-quarters of the length of the ladder.

3. A ladder, 16 ft. long, rests against a smooth vertical wall, making 30° with it. To prevent slipping, a rung of the ladder 4 ft. from the bottom is tied to the foot of the wall by a string. If the ground is smooth, find the tension in the string in terms of the weight of the ladder. [*Banaras*, 1955

4. The A-frame, as shown in the figure, is hinged at E and is supported on a smooth floor at A. The sides AC and CE make angles of $60°$ with the floor. The side $AB = DE = 1{\cdot}5$ m., and $BC = CD = 4{\cdot}5$ m. Find the horizontal and vertical components of the pin-reactions at B, C and D due to a load of 4000 kg., acting at the mid-point of the member BD. (The weight of the frame may be neglected.)



5. Two equal rods, AB and AC, each of weight W, are smoothly hinged at A and are placed in a vertical plane with their extremities B and C, resting on a smooth horizontal plane. They are kept from falling by a string, connecting the mid-points of the rods. If θ be the inclination of either rod with the vertical, find the tension in the string and show that the reaction at the hinge A is $W \tan \theta$ in the horizontal direction.

6. Two equal rods, AB and AC, each of weight W, are smoothly hinged at A and are placed in a vertical plane with their extremities B and C resting on a horizontal plane. They are kept from falling by two strings; one connecting end B and the foot of the perpendicular from B on AC and the other connecting C and the foot of the perpendicular from C on AB. If θ be the inclination of either rod with the vertical, show that tension in each string is

$$\frac{W \sin \theta}{4 \cos 2\theta},$$

and the reaction at A is $\frac{1}{4}W \tan 2\theta$.

7. Two light rods, AB and AC, each of length a are hinged at A and are placed in a vertical plane with the ends B and C, resting on a smooth horizontal plane. B and C are joined by a string of length a, and a weight W is fastened to the rod AC at a point D, at height h above the horizontal plane. Prove that the tension in the string is

$$Wh/3a.$$

8. A beam AB, weight W, is smoothly hinged to a wall at A and the other end B is fastened by a string which is attached to a point C on the wall, vertically above A at height h. The beam carries a load of nW at B. If the length of the string is l, show that the tension in the string is

$$(2n+1)\,\frac{l}{2h}\,W.$$

9. A door, weight 30 pounds, 6 ft. 8 in. high and 30 inches wide, is hinged along the left edge, 6 inches from the top and

from the bottom. Find the forces exerted by the door on the hinges, if the reaction at the upper hinge has no vertical component. [*Banaras*, 1952]

10. A gate, 2 m. high and $2\frac{1}{2}$ cm. wide, weighs 56 kg. and is supported by two hinges, 25 cm. from the top and the bottom respectively. The lower hinge can exert only a horizontal reaction. Find the reactions at both hinges, if a boy of weight 26 kg. is sitting on the end of the gate. [*U. P. E. S.*, 1964]

11. Two equal heavy spheres, of 3 cm. radius, are in equilibrium within a smooth spherical cup of radius 9 cm. Show that the action between the cup and one sphere is double of that between the spheres.

12. Three equal smooth cylindrical pencils, each of weight W, are tied together by a string and laid on a smooth table. The two lower pencils are pressing each other with a force P. Show that the tension in the string is $P+W/2\sqrt{3}$. [*Allahabad*, 1964]

13. A sphere, radius 12 inches and weight 300 pounds, is placed inside a hollow vertical cylinder of inside radius 16 inches; and over it is placed another sphere, radius 6 inches and weight 100 pounds. Find the reactions between the spheres, and between the spheres and the sides of the cylinder. [*Banaras*, 1962]

14. A hollow metal cylinder, open at both ends and of inside radius a, is placed on a horizontal plane, axis vertical. Inside it are placed two equal spheres of radius r, one above the other ($a < 2r < 2a$). If the weight of the cylinder be W and that of each sphere be W', show that the cylinder will stand upright without overturning if

$$2W'(a-r) \leqslant Wa.$$ [*Banaras*, 1937]

15. Two spheres, each of radius r and weight W, are placed inside a hollow vertical cylinder of inside radius a ($a < 2r < 2a$); the base of the cylinder is inclined at an angle α to the horizontal. Find the reactions between the base of the cylinder and the lower sphere, between the spheres and between each sphere and the side of the cylinder. Assume all surfaces to be smooth.

16. A light cord is loosely suspended from two points A and B on the same level and 10 metres apart. Four equal reflectors, each weighing 2 kg., are suspended from 4 points on the cords at distances such that the horizontal projections of the segments of the cord between the reflectors are equal. The lowest protion of the cord is horizontal and is 2·4 m. below the line joining A and B. Find the tension in each segment of the cord.

17. An equilateral triangular lamina ABC resting on a smooth horizontal plane is acted upon by a force of 5 lb. wt. along BC, 3 lb. wt. along AC and 2 lb. wt. along AD, where AD is per-

pendicular to BC. Find the force at B and the couple which will keep the lamina at rest. [*Jodhpur*, 1965]

18. ABC is a jib and tie arrangement (fig. 1). AB is the jib which is uniform and 10 ft. in length. The jib is hinged at A and the tie is secured to a point C, 10 ft. vertically above A. The jib is of weight 1 cwt. and carries a load of 1 ton at B. The angle BAC is 45°. Find the tension in the tie. [*Banaras*, 1952]

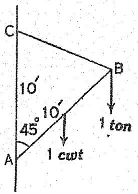


Fig. 1

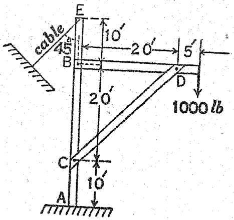


Fig. 2

19. Find the pull of the cable at E on the crane shown in Fig. 2. Find also the horizontal and vertical components of the reaction at A due to the 1000 lb. load. [*Agra*, 1957]

20. Find the magnitudes of the reactions of the pins at A, B and C on the member AC of the frame shown in the diagram (Fig. 3). [*Panjab*, 1957]

21. Two equal uniform rods, AB and AC are freely jointed at A. They are placed on two smooth pegs on the same level, the rod AB resting on one peg and AC on the other peg, having joint A above the line of the pegs. If a be the length of each rod and c be the distance between the pegs ($c < a$), show that in the position of equilibrium, the inclination of either rod with the horizontal is

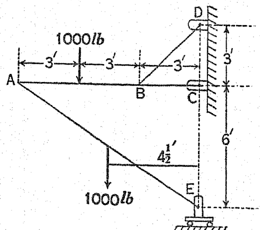


Fig. 3

$$\theta = \cos^{-1} (c/a)^{1/3}.$$

22. Two equal uniform rods AB and AC, each of weight W and length $2a$, are freely jointed at A. They are symmetrically placed over a smooth sphere of radius r. Show that the inclination θ of each rod to the horizontal is given by

$$r(\tan^3\theta+\tan\theta)=a,$$

and the reaction at A is

$$W\tan\theta.$$

[*Banaras*, 1964]

23. AB is a uniform rod of length $8a$, which can turn freely about the end A which is fixed; C is a smooth ring whose weight is twice that of the rod, which can slide on the rod and is attached by a string CD to a point D in the same horizontal plane as the point A; if AD and CD be each of length a, find the position of the ring and the tension in the string when the system is in equilibrium. Show that the action on the rod at the fixed end A is a horizontal force equal to $\sqrt{3}\ W$, where W is the weight of the rod.

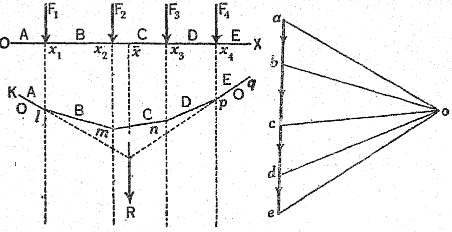
24. Two equal uniform rods, AB and BC, are smoothly hinged at B. The end A is hinged to a fixed point and to the free end C, a weightless smooth ring is attached which can slide along a smooth rod through A. If this rod AC be inclined at an angle α to the horizontal downward, show that, in the position of equilibrium, the angle between the rods AB and BC, is

$$2\tan^{-1}(2\tan\alpha).$$

CHAPTER V

GRAPHICAL METHODS. FRAMES

5·1. Resultant of Coplanar Parallel Forces. Let F_1, F_2, F_3 and F_4 be four parallel forces, atting at the points x_1, x_2, x_3 and x_4 of a line OX. Indicate the forces in Bow's notation by AB, BC, CD and DE, as shown in fig. (a).



(a) (b)

The magnitude of the resultant. Draw vectors $\overline{ab}$, $\overline{bc}$, $\overline{cd}$, $\overline{de}$ to some scale to indicate the given forces in the force diagram (fig. b). The closing line $\overline{ae}$ of the force polygon $abcde$ will give the magnitude of the resultant R. Since the forces are all parallel, the force polygon for these will be a line.

The position of resultant in the space diagram. Take any point o in the force diagram. Join ao, bo, co, ... This figure is called a *polar diagram.* The point o is called a *pole* and the lines ao, bo, co, ... are called *rays* of the polar diagram (fig. b).

Starting from any point l on the line of action of force AB, draw lines kl, lm, mn, ... parallel to the corresponding rays ao, bo, co, ... and cutting the lines of action of F_1, F_2, ... in l, m, ... The diagram $klmnpq$ is

called a *funicular polygon* (fig. *a*) and *kl*, *lm*, *mn*, ... are called *strings* of the funicular polygon. Produce the two free strings *kl* and *pq*. The meeting point of these will give the position of R as shown in fig. (*a*).

PROOF. From the triangle of forces *abo*, we have

$$\overline{ab} = \overline{ao} + \overline{ob}.$$

But $\overline{ab} = F_1$; therefore

the force F_1 acting at l

$$= \overline{ao} \text{ along } kl + \overline{ob} \text{ along } ml.$$

Similarly,

the force F_2 acting at $m = \overline{bc}$

$$= \overline{bo} \text{ along } lm + \overline{oc} \text{ along } nm;$$

the force F_3 acting at $n = \overline{cd}$

$$= \overline{co} \text{ along } mn + \overline{od} \text{ along } pn;$$

the force F_4 acting at $p = \overline{de}$

$$= \overline{do} \text{ along } np + \overline{oe} \text{ along } qp.$$

When we add these up, all the forces on the right-hand side with the exception of the first and the last, cancel each other, as they consist of pairs of equal and opposite forces. We get

the forces $F_1 + F_2 + F_3 + F_4$

$$= \overline{ao} \text{ along } kl + \overline{oe} \text{ along } qp$$

$$= \overline{ae},$$

the line of action being through the point of intersection of the lines *kl* and *qp*.

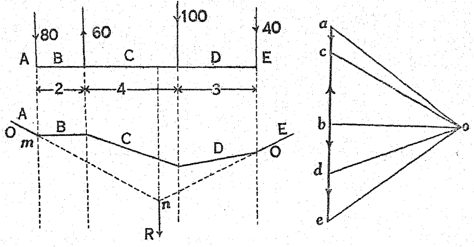
If the force polygon closes, i.e., the final point e coincides with the initial point a and the two free strings *kl* and *pq* are two parallel lines, the resultant of the system is a couple of moment

$$M_\delta = ao \, . \, \delta,$$

where δ is the perpendicular distance between the two free strings *kl* and *pq*.

NOTE : The advantage of using Bow's notation is that the forces *AB*, *BC*, ... in the space diagram may be denoted by the corresponding lower case letters *ab*, *bc*, ... in the force diagram. Furthermore, the strings *kl*, *lm*, ... which are parallel to *ao*, *bo*, ... may be denoted by *AO*, *BO*, ... in the funicular polygon using Bow's notation.

Ex. 1. Find the magnitude, direction and the position of the resultant of the four forces shown in fig. (*a*). The forces are in kilograms and the distances shown are in metres.



(*a*) (*b*)

The space diagram (fig. *a*) is given, showing the lines of action of the forces. These four forces are denoted by *AB*, *BC*, *CD* and *DE* in Bow's notation. Four vectors *ab*, *bc*, *cd* and *de* are drawn to scale (1 cm. = 40 kg.) as in fig. (*b*). Thus vector *ae* gives the resultant in magnitude and direction. A point *o* is selected and rays *oa*, *ob*, ... are drawn.

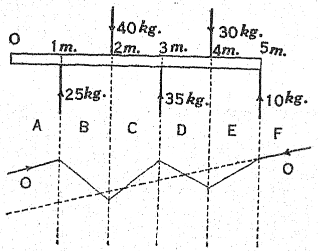
Starting from a point *m* on force *AB*, the string *AO* is drawn parallel to the ray *ao*; then strings *BO*, *CO*, *DO* and *EO* are drawn parallel to the corresponding rays in the force diagram. The intersection of the free strings *AO* and *OE* at *n* determines the line of action of the resultant force.

By measurement the magnitude of $R = 160$ kg. (length of vector *ae*) and its distance from 80 kg. force = 5·25 m.

Ex. 2. A bar 5 m. long is acted on by the forces shown in the figure. on the next page. Find the resultant force acting on the bar.

The space diagram is given. Extend the lines of action of all the forces. Draw the force polygon* (scale 1 cm. = 10 kg.). It is found that it closes, i. e., *f* coincides with the initial point *a*. Select a point *o* and draw rays *ao*, *bo*, *co*, ... Draw the funicular

polygon in the space diagram, by drawing strings AO, BO, ... parallel to rays ao, bo, ... The resultant is a couple of moment 40 kg.-m.



5·2. Equilibrium of parallel forces. The resultant of a coplanar parallel force system, in general, is a single force R. If, however, the force polygon closes, the magnitude of R is zero, and the resultant is a couple C (§ 5·1). If the funicular polygon also closes, i.e., the two free strings are collinear, the moment of the couple C is zero, and the force system will be in equilibrium.

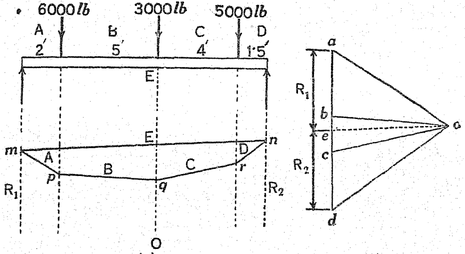
Conversely, if the parallel force system be in equilibrium, the force polygon and the funicular polygon must both close.

By the application of these two conditions, two unknown forces occurring in a system in equilibrium may be determined.

Ex. The horizontal beam AB of length 12·5 ft. (fig. (*a*) p. 63) carries concentrated loads of 6000 lb., 3000 lb. and 5000 lb. at 2 ft., 7 ft. and 11 ft. from the left end, and is supported at its two ends. Find the reactions on the beam at the supports.

The space diagram is drawn to scale and the loads indicated in Bow's notation by AB, BC, CD. It is required to determine the reactions DE and EA at the supports. The five forces AB, BC, CD, DE and EA are in equilibrium. The force polygon (fig.(*b*) p. 63). is drawn to scale (1″ = 12000 lb.) for the known vectors ab, bc, cd. Any convenient pole o is selected and rays ao, bo, co, do are drawn. Funicular polygon $mpqrn$ is drawn with sides parallel to the corresponding rays. Let the free strings AO

and DO meet the lines of action of the reactions at m and n as shown in fig. (*a*).

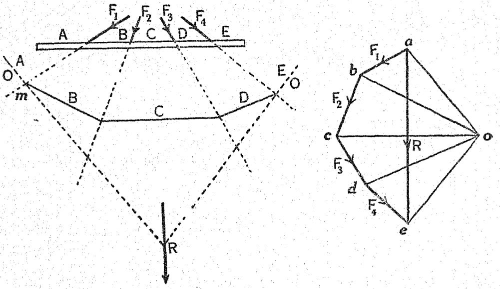


(*a*) (*b*)

Join mn which is the closing line of the funicular polygon.

From o draw oe parallel to mn to meet ad in e (fig. *b*). Then $\overline{de}$ and $\overline{ea}$ represent the two reactions. By measurement their magnitudes are $ea = R_1 = 6950$ lb. and $\overline{de} = R_2 = 7050$ lb.

5·3. Resultant of Coplanar Non-concurrent Forces. The resultant of a system of coplanar non-concurrent forces can be found by the construction



(*c*) (*d*)

of the force diagram and the funicular polygon, as is done in the case of the parallel force system.

Let the system consist of forces F_1, F_2, F_3 and F_4, represented in Bow's notation by AB, BC, CD and DE, as in fig. (c). The force polygon (fig. d) is then drawn to scale, representing the given forces by vectors $\overline{ab}$, $\overline{bc}$, $\overline{cd}$ and $\overline{de}$. The closing line $\overline{ae}$ of the force polygon gives the magnitude and direction of the resultant R.

Taking a point o as pole (fig. d), rays ao, bo, co, do and eo are drawn. Starting from any point m on the line of action of force AB, draw strings AO, BO, CO, DO and EO, parallel to the corresponding rays, in the space diagram and obtain a funicular polygon. The intersection of the two free strings AO and OE will give the point through which the resultant R of the system acts. The proof can be given on the same lines as in the case of parallel forces.

In case the force polygon closes and the two free strings AO and OE in the funicular polygon are parallel, the resultant of the system is a couple C. The perpendicular distance between the parallel free strings is the arm of the couple, and the ray ao or oe in the force diagram gives the magnitude of a force of the couple. Therefore the moment of the resultant couple C

$$= ao \times \text{perpendicular distance between strings } AO \text{ and } OE.$$

In case the force polygon closes and the funicular polygon closes, i.e., the two free strings are collinear, there is no resultant and the system is in equilibrium.

NOTE. The main difference between the parallel force system and non-concurrent force system is that for the former the force polygon *abcde* is a line and for the latter the force polygon *abcde* is some closed figure.

5·4. Equilibrium of Coplanar Forces. It is seen from the preceding article that the necessary and sufficient conditions of equilibrium are :—

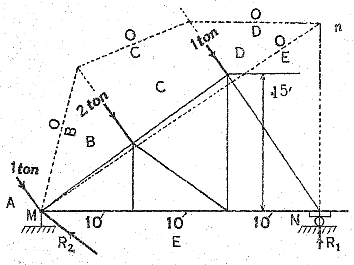
(1) the force polygon must close,

(2) the funicular polygon must close.

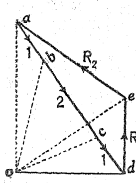
Making use of these conditions, an equilibrium problem can be solved.

The closing of the force polygon gives two independent conditions of equilibrium (corresponding to $\Sigma X_1=0$ and $\Sigma Y_1=0$), and the closing of the funicular polygon gives a third independent condition (corresponding to $\Sigma M_0=0$). So unknown elements, not exceeding three, can be determined, viz. (*a*) the magnitudes of three forces, directions being known or (*b*) one force completely and magnitude of another force, the direction of which is known.

Ex. Fig. (*a*) below represents an unsymmetrical roof truss. Three wind loads of 1 ton, 2 tons and 1 ton are acting at the joints perpendicular to the surface of the roof. The truss is supported by a pin bearing at M and on rollers at N. Find the reactions.



(*a*)



(*b*)

The space diagram is given in fig. (*a*). Since the truss rests on rollers at N, the direction of reaction R_1 is vertically upwards. The reaction at pin bearing M is R_2 whose direction and magnitude are not known. Indicating the forces by Bow's notation, we have five forces AB, BC, CD, DE and EA in equilibrium.

The known forces represented by the vectors *ab bc* and *cd* are drawn in a force polygon to scale ($1''=3$ tons) as in fig. (*b*). Through *d* draw a vertical line *de*. Since the magnitude of force DE is not known, the position of *e* remains to be found.

Select a pole *o* and draw the rays *ao*, *bo*, *co* and *do*. Nothing is known about the reaction at M, except that it passes through M. Since M is the only known point on the line of action of force EA, the funicular polygon must be started from M. Draw the string BO parallel to ray *bo*, the string CO parallel to the ray *co*, and finally the string DO parallel to ray *do*, meeting force DE at *n* (fig. *a*). Since the system is in equilibrium, the funicular polygon must close.

Hence join Mn, which is the closing line. The missing ray eo must be parallel to this closing line. So the point e can be located by finding the intersection of oe (parallel to closing line Mn) with the vertical line previously drawn through d. Join ae.

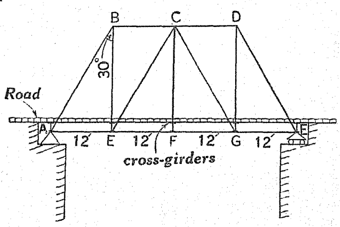
Since both the forces EA and AB meet at M, hence the string corresponding to the ray oa need not be considered. (In fact it reduces to a string of zero length starting from M and finishing at M.) In the force diagram force EA is represented by the vector ea, in magnitude, direction and sense, and force DE is represented by vector de.

The magnitudes of the two reactions, as found by scaling the lengths, are

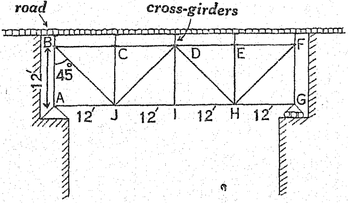
$$R_1 = de = 1{\cdot}7 \text{ tons}$$

$$R_2 = ea = 2{\cdot}75 \text{ tons at } \angle\theta_x = 146{\cdot}5°.$$

5·5. Simple Framed Structures. The sketches given below illustrate some simple framed structures known as *plane trusses.* A plane truss is defined as a system of straight bars jointed together at their ends in such a way as to form a rigid framework in one plane. The straight bars are called *members* of the truss. To ensure rigidity they are connected in such a way as to divide the enclosed space within the truss into triangular spaces.

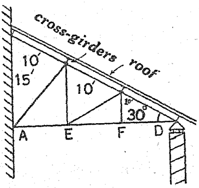


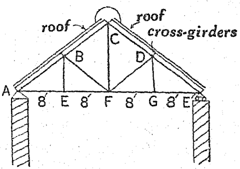
The members are either pin jointed by means of cylindrical pins or rivetted by help of gusset plates. Those joints which are situated at the perimeter of the truss are called panel points. The loads are generally applied at the panel points, as would be seen from the above figures.



Structures are supported on abutments or piers called *supports.* One end of the truss is set upon the support and fastened to it

(the point A in each figure). The other end is supported in such a manner that expansion or contraction due to change of tempera-





ture etc., may be taken up. For this purpose the expansion end is mounted on rollers.

Examples of coplanar force systems in equilibrium are met in the analysis of the stresses in the members of the trusses. To avoid bending of the members the external load must be applied only at the joints.

Trusses are designed for carrying loads that may act upon the structure. The permanent forces on the trusses due to road way flooring, weight of members, etc., are called *dead-load.* Other forces on the trusses, such as effect of wind, vehicles, traffic etc., which are variable, are called *live-load.* The maximum value of live load should be foreseen and estimated. Each member of the truss is then designed for the stress resulting from the combination of the loads.

After the maximum loads have been estimated, the stress in the members are calculated, taking into consideration the following assumptions—

(*a*) The external loads on the truss act only at the joints. It is often done so practically. Thus the weight of the roadway or the roof carried by trusses in above figures are thrown on the panel points by the use of cross beams that rest on the truss only at the joints.

(*b*) The weight of the members is neglected as compared to other loads, or the weight of a uniform member is replaced by two parallel forces, each equal to half the weight and acting at the joints.

These assumptions give us a frame-work of weightless rigid members, with forces acting at the joints. Since the forces acting at each of the two joints of a member will reduce to a single resultant, therefore, for equilibrium the resultant at one end of a member is equal and opposite to the resultant at the other end.

Thus, every member is a two-force piece and is either in tension or compression. A member in tension is called a *tie*; a member in compression is called a *strut*.

5·6. Method of Joints. Before determining the stresses in the members of a truss all the external forces, including reactions at the supports, should be determined considering the whole truss as a free-body in equilibrium. There are two methods to find the internal stresses in the members.

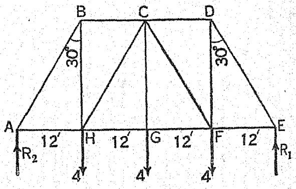
In the first method, known as the *method of joints*, each joint in turn is taken as a free-body. The forces, due to stresses in the members and external load (if any) at a joint constitute a coplanar concurrent force system in equilibrium and a solution is obtained from the equations of equilibrium

$$\Sigma X_1 = 0,\ \Sigma Y_1 = 0.$$

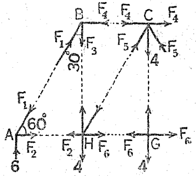
Two unknown stresses can be found from these equations. Therefore, the joints should be taken in such an order that there are not more than two unknown stresses at each stage.

In many cases the sense of the unknown forces in a member is evident by inspection. When this is not the case, the force may be assumed to be directed away from the joint. If after solving the equations of equilibrium the sign of the force is found to be negative the sense of the force must be opposite to that assumed.

Ex. The figure below, represents a bridge truss. The panel lengths are each equal to 12 ft. The load carried by panel points F, G and H, is 4 tons each. Find the stresses in all the members.



Considering the whole truss as a free-body, the reactions at supports A and E are equal, i.e. $R_1 = R_2 = 6$ tons (as loads and figure are both symmetrical). Since the structure and loading are symmetrical, the stresses, in the members on the right of CG are same as those in the correspond-



ing members on the left. Therefore it is necessary to consider free-body diagrams for the points A, B, C, G and H only. From the free-body diagram for the point A, we have

$$\Sigma X_1 = F_2 - F_1 \cos 60° = 0.$$
$$\Sigma Y_1 = -F_1 \sin 60° + 6 = 0.$$
$$\therefore \begin{cases} F_1 = 6 \times 2/\sqrt{3} = 6{\cdot}92 \text{ tons.} \\ F_2 = 6{\cdot}92 \times \tfrac{1}{2} = 3{\cdot}46 \text{ tons.} \end{cases}$$

Next, the free-body diagram for the point B gives:

$$\Sigma X_1 = F_1 \sin 30° - F_4 = 0. \qquad \therefore \quad F_4 = 6{\cdot}92 \times \tfrac{1}{2} = 3{\cdot}46 \text{ tons.}$$
$$\Sigma Y_1 = F_1 \cos 30° - F_3 = 0. \qquad \therefore \quad F_3 = 6{\cdot}92 \times \sqrt{3}/2 = 6 \text{ tons.}$$

Next consider the joint H. We get

$$\Sigma X_1 = F_6 - F_5 \cos 60° - F_2 = 0.$$
$$\Sigma Y_1 = -F_5 \sin 60° + F_3 - 4 = 0.$$
$$\therefore \quad F_5 = (6-4)2/\sqrt{3} = 2{\cdot}308 \text{ tons.}$$
$$\therefore \quad F_6 = 3{\cdot}46 + 2{\cdot}308/2 = 4{\cdot}614 \text{ tons.}$$

Then, joint G gives

Stress in member $CG = 4$ tons.
Stress in member $GF = F_6 = 4{\cdot}614$ tons.

Lastly, joint C gives

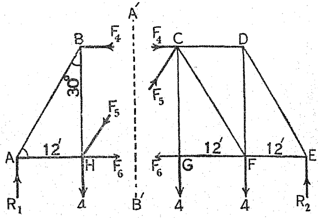
$$\Sigma Y_1 = 2F_5 \cos 30° - 4 = 0. \quad \therefore \quad F_5 = 2 \times 2/\sqrt{3} = 2{\cdot}308 \text{ tons.}$$

which gives a check that the results obtained are correct.

To sum up

Members	Stress in tons	Nature
AB & DE	6·92	Compression
AH & EF	3·46	Tension
BC & CD	3·46	Compression
CH & CF	2·3	Compression
HG & FG	4·6	Tension
BH & DF	6·0	Tension
CG	4·0	Tension

5·7. Method of Sections. In determining the stresses in the members by the *methods of sections*, the truss is divided into two parts by a section so that it cuts not more than three members. The part of the truss on either side of the section is then treated as a free-body in equilibrium under the action of the known external loads (acting on that part) and the stresses exerted by the members which have been cut, on the part considered. It will be seen from the figure that the method of sections gives a coplanar non-concurrent force system in equilibrium, and the solution is, therefore, obtained from the equations



$$\Sigma X_1 = 0,$$
$$\Sigma Y_1 = 0,$$
$$\Sigma M_A = 0.$$

Not more than three unknown stresses can be determined from these equations. Consequently, the section must not cut more than three members in which the stresses are unknown.

The method of sections can be advantageously used when we require the stress in a particular member of a truss, and not in the others. By taking a section cutting that member we can find that stress directly. If we apply the method of joints we shall have to start from one end and find the stresses in all the intervening members before reaching that particular member.

Ex. Find the stresses in the members BC, CH and HG of the bridge-truss of the preceding example by the method of sections.

As before, $R_1=6$ tons. Consider a section along the line $A'B'$ (fig. p. 70). The part of the truss, on the left, is a free-body

in equilibrium under forces $R_1=6$ tons, load $=4$ tons, and stresses F_4, F_5 and F_6. Therefore

$$\Sigma X_1 = F_6 - F_4 - F_5 \cos 60° = 0. \qquad . \quad . \quad . \quad (1)$$

$$\Sigma Y_1 = 6 - 4 - F_5 \sin 60° = 0. \qquad . \quad . \quad . \quad (2)$$

$$\Sigma M_H = F_4 \times BH - 6 \times AH = 0. \qquad . \quad . \quad . \quad (3)$$

From (3), $F_4 = 6 \times \dfrac{AH}{BH} = 6 \tan 30° = \dfrac{6}{\sqrt{3}} = 3{\cdot}46$ tons.

From (2), $F_5 = \dfrac{2}{\sin 60°} = \dfrac{4}{\sqrt{3}} = 2{\cdot}308$ tons.

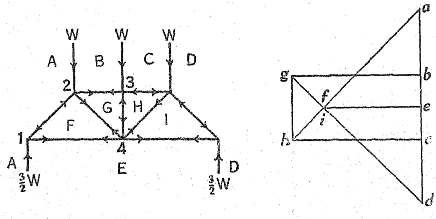
From (1), $F_6 = 3{\cdot}46 + \dfrac{4}{\sqrt{3}} \times \dfrac{1}{2} = 4{\cdot}614$ tons.

5·8. Graphical Method. The graphical method of analysis of stresses in the members of framed structures is simpler than the analytical method and is widely used by engineers, particularly when the form of the structure is complicated and the load unsymmetrical. The stresses found by the graphical method are not so accurate as those found by the analytical method.

As explained earlier the system of forces due to the members and the loads may be considered as sets of concurrent forces in equilibrium acting at each joint. Therefore the force diagram pertaining to any joint is a closed force polygon and can be drawn as explained in § 5·3.

Let us consider a simple structure as shown in fig. (*a*), on page 72, loaded with three equal weights W. Taking the entire structure as a free-body, indicate external forces (including reactions) by Bow's notation in clockwise sequence and then draw a force polygon (adopting some suitable scale for forces) as shown in fig. (*b*). The force polygon is a closed line *abcdea*, since the forces acting on the structure are parallel.

Next each triangular space within the structure is also denoted by letters in continuation as shown in



(*a*) (*b*)

fig. (*a*). Then each joint is taken (in the order indicated by the numbers 1, 2, 3, ...) as a free-body, and its force polygon is superimposed on the original diagram as shown in fig. (*b*). Thus the linked polygons pertaining to various joints, plotted together, form a stress diagram for the truss (fig. *b*).

Referring to the stress diagram it would be seen that

For joint no. 1 the force polygon is *eafe*

For joint no. 2 the force polygon is *fabgf*

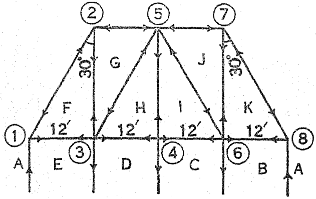
For joint no. 3 the force polygon is *bchgb*

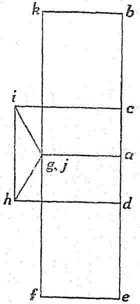
For joint no. 4 the force polygon is *efghie*

It is necessary to take the forces in a sequence (either clockwise or anticlockwise) throughout the solution. The anticlockwise sequence would yield a diagram which would be the exact reverse of fig. (*b*).

Since for a concurrent force system not more than two elements can be determined, hence it is necessary that joints should be selected in such an order that not more than two unknown stresses are introduced at each stage.

Ex. 1. Find the stresses in the members of the bridge truss of Ex. § 5·6, graphically. The loads, carried by the panel points F, G and H are 4 tons each.





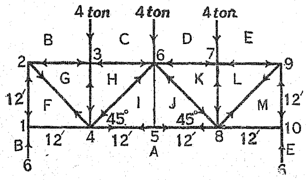
Indicate the forces by Bow's notation AB, BC, CD, DE and EA, and in continuation indicate the triangular spaces by F, G, H, I, J and K. First draw the force polygon for the external forces to scale ($\frac{3}{4}'' = 4$ tons) which is a line *abcdea*. Next start from the joint (1) and draw the force triangle *eaf* (by drawing *ef* and *af* parallel to the corresponding forces, through *e* and *a* respectively).

Then for joint (2) draw the force triangle *fag*;
for joint (3) draw the force polygon *fghdef*;
and for joint (4) draw the force polygon *dhic*.
The rest can be drawn by symmtery.

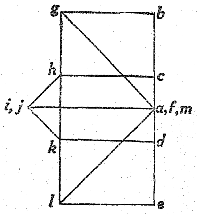
The stresses in the members, as obtained by measurement, are given in the table below :

Member	Stress	Member	Stress
1-2 & 7-8	6·9 tons (C)	3-4 & 4-6	4·6 tons (T)
1-3 & 6-8	3·45 tons (T)	3-5 & 5-6	2·2 tons (C)
2-3 & 6-7	6·0 tons (C)	4-5	4·0 tons (T)
2-5 & 5-7	3·45 tons (T)		

Ex. 2. Find graphically the stresses in the members of the bridge truss shown below. Loads of 4 tons each act at three panel points as shown in the figure. [*Allahabad*, 1964]



(*a*)



(*b*)

Indicate the forces by Bow's notation in a clockwise sequence and the triangular spaces by letters in continuation.

Draw the line polygon *abcdea* for the external forces (scale 1″ = 8 tons).

Starting from the joint 1 draw force polygons pertaining to each joint, linked to each other, and obtain a stress diagram for the bridge truss (fig. *b*).

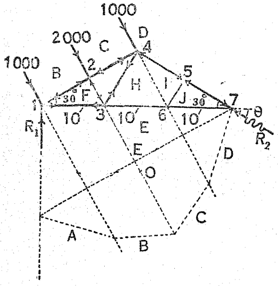
The stresses in the members, as obtained by measurement, are

Member	Stress	Member	Stress
1-2 & 9-10	6 tons (C)	4-5 & 5-8	8 tons (T)
2-3 & 7-9	6 tons (C)	3-4 & 6-7	4 tons (C)
3-6 & 6-7	6 tons (C)	4-6 & 6-8	2·8 tons (T)
2-4 & 8-9	8·5 tons (T)	1-4, 8-10, 5-6	Zero

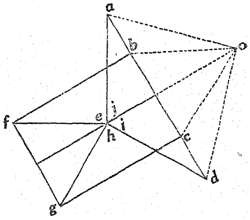
Ex. 3. A roof truss, shown in figure (*a*) on the next page is subjected to wind load as indicated, the loads being perpendicular to the upper chord. The truss rests on a smooth plate at the left end so that the reaction at that end is vertical. Find the magnitude and direction of the reaction at the right end of the truss and the stresses in the members.

First taking the entire truss as a free-body in equilibrium, we find the reaction of the supports R_1 and R_2, graphically. To determine them draw known vectors *ab*, *bc* and *cd*, to scale $\frac{3}{4}''$ = 2000 lb. giving designation to forces by Bow's notation. Select any

point o as pole and draw rays ao, bo, co and do. Since the line of action of R_2 is not known, hence start the funicular polygon



(a)



(b)

from the right end support (a point known to be on R_2). Draw strings DO, CO, BO and AO. The closing line of the funicular polygon will determine R_1 and R_2. Thus, draw line eo parallel to the closing line EO and draw ae vertical, and obtain the intersection point e. The vectors de and ea will give the magnitudes and directions of the reactions R_2 and R_1.

Next draw linked force polygons for each joint, in the order indicated by the numbers 1, 2, 3, ... taking the forces in clockwise sequence. This will give a complete stress diagram for the truss (fig. b).

By measurement, the stresses are:

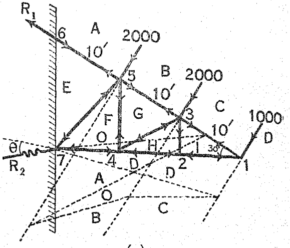
Member	Stress	Member	Stress
1-2	2800 lb. (C)	1-3	2000 lb. (T)
2-4	2800 lb. (C)	2-3	2000 lb. (C)
4-5	2300 lb. (C)	3-4	2000 lb. (T)
5-7	2300 lb. (C)	Others	Zero.

The reaction R_1 at the left support $=2300$ lb. vertically upwards.

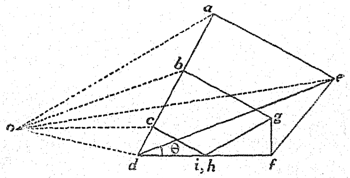
The reaction R_2 at the right support $=2300$ lb. inclined at 150° with the horizontal.

Ex. 4. Find the stresses in the members of the cantilever roof truss, when loaded at its upper chord panel points as shown in fig. (*a*).

First taking the entire truss as a free-body, determine the reactions R_1 and R_2 of the wall at points 6 and 7, graphically. Indicating forces by Bow's notation, draw vectors *ab*, *bc*, *cd* to scale (1″=4000 kg.). Select any point *o* as pole and draw rays *ao*, *bo*, *co* and *do*. Starting from point 7 of the space diagram, (the only point known on the line of action of R_2) draw strings *DO*, *CO*, *BO* and *AO*, and obtain the closing line *EO* of the funicular polygon. Coming back to the force diagram, draw through *o* the ray *oe* parallel to the closing line and through *a* draw line *ae* parallel to force *AE* and obtain the intersection point *e*. Join *de*. Then vector *de* represents the reaction R_2 at 7 both in magnitude and direction and *ea* represents magnitude of force R_1. By measurement, R_2 is 6450 kg. at an $\angle 21°$ with x-axis and R_1 is 4100 kg.



(*a*)



(*b*)

Next taking the joints in the order 1, 2, 3, ... draw force polygons for each joint, and obtain a complete stress diagram for the truss.

By measurement the stresses are :

Member	Stress	Member	Stress
1-2	2100 kg. (C)	3-5	3000 kg. (T)
1-3	1800 kg. (T)	4-5	1100 kg. (T)
2-3	Zero	5-6	4100 kg. (T)
2-4	1800 kg. (T)	5-7	3000 kg. (C)
3-4	2200 kg. (C)		

Examples 7

1. Two forces of 50 kg. and 75 kg. act on the beam *AB*, as shown in fig. 1. Determine graphically the reactions at the supports *A* and *B*.

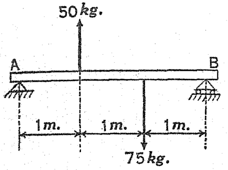


Fig. 1

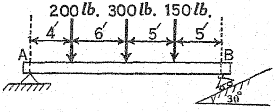


Fig. 2

2. Determine graphically the reactions at the supports *A* and *B* of the beam *AB* loaded as shown in fig. 2.

3. Find the forces in the members of the derrick crane shown in fig. 3, loaded at *A* with 10 tons. (*BC* is vertical and the members are pin-jointed at *A*, *B*, *C* and *D*.) [*Roorkee, Arch.*, 1966]

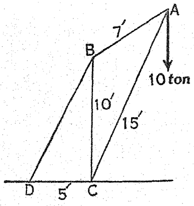


Fig. 3

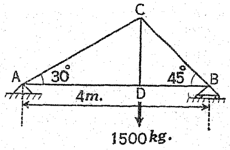


Fig. 4

4. A simply supported roof truss, shown in fig. 4, carries a load of 1500 kg. at *D*. The span *AB* of the truss$=4$ m. and the angles at *A* and *B* are 30° and 45°. Find the reactions at the supports and the stress and its nature in each member of the truss.

5. The roof truss, shown in fig. 5, is supported at points *A* and *E*. It carries loads of 4 metric tonnes each, at points *B*, *C* and *D*. The span $AE=20$ ft., and the angles at *A* and *E* are 45°.

The pieces BF and FD are perpendicular to AC and CE respectively. Find the stress in each member.

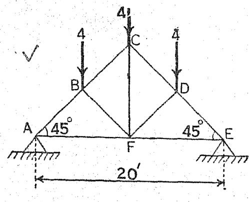


Fig. 5

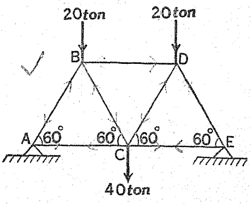


Fig. 6

6. Find the forces in the members of the frame loaded as shown in fig. 6, BD being parallel to AE. [*Panjab*, 1956]

7. The truss, shown in fig. 7, carries loads of 4 tons and 7 tons at points B and C. It is supported at the end points A and D. The lengths of the members AB, BC, CD, EF, BF and CE are each 5 ft. Find the stress in each member of the truss.

8. The warren truss, shown in fig. 8, is simply supported at A and E. It carries loads of 2000 kg. each, at panel points G and F. The piece $AG=GF=FE=2$ m. and the angles of the triangles are each 60°.

Draw the force diagram for the truss and determine the stress in each member of the truss.

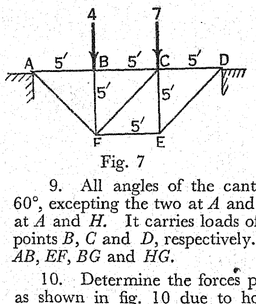


Fig. 7

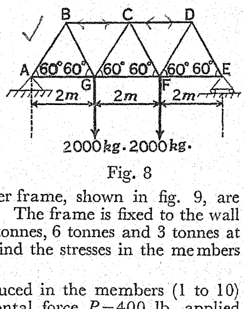


Fig. 8

9. All angles of the cantilever frame, shown in fig. 9, are 60°, excepting the two at A and H. The frame is fixed to the wall at A and H. It carries loads of 6 tonnes, 6 tonnes and 3 tonnes at points B, C and D, respectively. Find the stresses in the members AB, EF, BG and HG.

10. Determine the forces produced in the members (1 to 10) as shown in fig. 10 due to horizontal force $P=400$ lb. applied

at the top. In the tower BC, DE, FG are parallel and $AB = BD = DF$. [*Agra*, 1957]

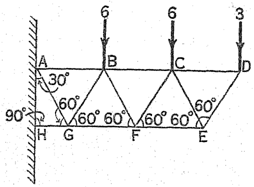


Fig. 9

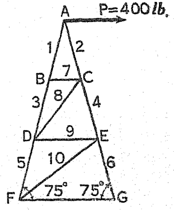


Fig. 10

11. The frame, shown in fig. 11, is attached to a wall at A and E and carries two loads of 4000 kg. and 2000 kg. at points B and C. The pieces AB, BC, CD and BD are 5 ft. each; and angle DBE is 60°. Draw the force diagram for the truss, and determine the magnitude and the nature of stresses in all the members of the frame.

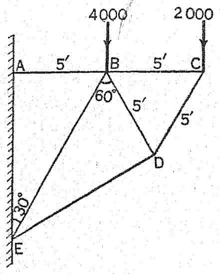


Fig. 11

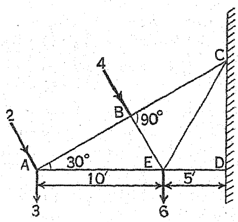


Fig. 12

12. A cantilever frame, as shown in fig. 12, is attached to a wall at C and D. It carries loads of 2 tons and 4 tons at A and B, perpendicular to AB, and vertical loads of 3 tons and 6 tons at A and E. The piece AE is 10 ft., ED is 5 ft., the angle CAD is 30° and the piece BE is perpendicular to AC. Find the forces and the nature of stress in all the members of the truss.

CHAPTER VI

FRICTION

6·1. Definition and general considerations. When a body in contact with another body slides or tends to slide over it, then a force of resistance, other than the normal reaction, comes into play which acts tangential to the surface of contact, and opposes the relative motion. This resisting force is called the *force of friction.* Friction is of importance in mechanics. We often recognise it as a waste of energy and a source of loss in certain machines, and attempts are made to reduce it as much as possible by the help of lubricants. There is, however, another aspect. Friction is often indirectly the means of producing motion. Consider the motion of a car. The wheels are pressed against the ground due to its weight. There is friction enough to prevent the wheels from slipping. Consequently when a turning force is applied to the wheels by the engine of the car, the wheels are made to roll without slipping. Friction is also a desirable element for various kinds of brakes, friction drives, etc.

The laws governing friction between two dry surfaces are quite different from those for two surfaces separated by a lubricant. The former were first given by the French engineer Charles Coulomb (1736-1806). Dry friction is caused by the interlocking of the microscopic irregularities of the surfaces in contact and the resulting plastic deformation when relative motion ensues.

Dry friction is a passive force. It is called into play only when there is a tendency of relative motion between two bodies. Its magnitude is just sufficient to preserve equilibrium and prevent motion. If the force P which tends to cause motion increases, the frictional force also increases. However, there is a limit to the amount of friction which can be called into play.

When this limit is reached the friction is called *limiting friction.* If the applied force P increases still further, the equilibrium will be disturbed and motion will ensue.

It is found as a result of experiments that

(i) The limiting friction is independent of the magnitudes of the areas in contact.

(ii) The limiting friction between two bodies bears a constant ratio to the normal reaction between them. This ratio is called the *coefficient of friction.* Thus, if R denotes the normal reaction, then the limiting friction is μR, where μ, the coefficient of friction, is a constant depending on the nature of the materials in contact.

(iii) When the motion takes place the laws of limiting friction still hold, but the coefficient of friction μ is slightly less than that for the static case. To distinguish between the two cases we respectively call them the coefficient of *static friction* and the coefficient of *kinetic friction.*

The value of μ depends upon the materials and also on the roughness of the surfaces in contact. Some of the values are given below.

Materials	*Coefficient of static friction*
wood on wood	0·25-0·5
metal on wood	0·2 -0·6
metal on metal	0·15-0·25
leather on wood	0·25-0·5
leather on metal	0·3-0·6

The friction between two surfaces completely separated by a film of lubricant, depends on the properties of the lubricating fluid, and is better studied under fluid mechanics. Fluid friction comes into play only during motion, and depends on the viscosity of the fluid, the thickness of the film, the area in contact and the relative velocity of the two surfaces. It is independent of the normal reaction between the two surfaces. Most of these properties are completely at variance with the laws of dry friction.

So the friction between partially lubricated surfaces may not obey any fixed laws.

A third type of friction called rolling friction comes into play when a sphere or a cylinder rolls over a surface. This friction arises on account of the slight deformation of the two surfaces in contact, and is much less in magnitude than sliding friction. Consequently a number of machines use ball or roller bearings in place of axle bearings in order to reduce friction. The laws of rolling friction are not well established, and we shall not consider them here.

6·2. Angle of friction. When a body has a tendency to slide over a surface under the action of external forces, the action of the surface on the body consists of two forces:

(i) the normal reaction R, acting along the normal to the surface, and

(ii) the frictional force F, acting along a tangent to the surface.

Therefore the resultant reaction of the surface on the body, is

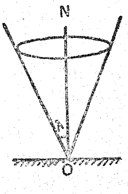
$$S=\sqrt{(R^2+F^2)},$$

making an angle $\tan^{-1}(F/R)$ with the normal.

This angle increases as F increases, and the maximum value is reached when the friction is limiting. In this case the angle

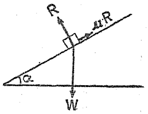
$$=\tan^{-1}(\mu R/R)=\tan^{-1}\mu=\lambda, \text{ say.}$$

This angle which the resultant reaction makes with the normal in the case of limiting friction is called the *angle of friction*.



If we draw a cone of semi-vertical angle λ with the normal ON to the surface as axis and O as vertex, then the resultant reaction at O will always lie within this cone, and in the limiting case will lie on the surface of the cone. This cone is called the *cone of friction*.

6·21. Angle of repose. If a body rests on an inclined plane and if α, the inclination of the plane, is such that the body is on the point of moving down the plane due to its weight, angle α is called the *angle of repose*.



The free-body diagram for the body gives three forces R, μR and W in equilibrium. Resolving the forces along and perpendicular to the plane, we get

$$\mu R = W \sin \alpha \text{ and } R = W \cos \alpha.$$

Therefore, by division,

$$\tan \alpha = \mu = \tan \lambda,$$

or

$$\alpha = \lambda.$$

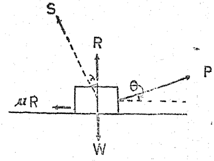
Hence the angle of repose = the angle of friction.

This gives a convenient method to determine the coefficient of friction experimentally. A body is placed on an inclined plane and then the inclination is slowly increased until the body is about to move down the plane. Then the coefficient of friction μ is equal to the tangent of the angle of repose.

6·3. Problems on Friction. When a body is at rest on a rough surface, then in considering its equilibrium the force of friction F should also be taken along with the other forces acting on the free body. The direction of F will be opposite to that in which the body will move if there was no friction. The problem is then solved making use of the equations of equilibrium for coplanar forces. If the body is just on the point of moving, then the frictional force F should be taken equal to the limiting friction μR.

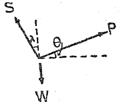
Ex. 1. Find the least force required to move a body on a rough horizontal plane, and the angle this force makes with the horizontal plane. [*Roorkee*, 1965]

Let W be the weight of the body on which a force P, acting at an angle θ with the horizontal, just causes the body to move. The other two forces acting on the body are the reaction R and the friction μR.



To simplify the solution, let us combine R and μR into the resultant reaction S acting at an angle λ with the vertical. Then the three forces W, P and S are in equilibrium.

Applying Lami's theorem, we have :



$$\frac{P}{\sin(180-\lambda)}=\frac{W}{\sin(90+\lambda-\theta)}$$

or $$P=\frac{W\sin\lambda}{\cos(\lambda-\theta)}.$$

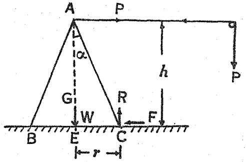
For minimum value of P, $\theta=\lambda$, i.e., P should be perpendicular to the resultant reaction S. Also,

$$P_{\min}=W\sin\lambda.$$

Ex. 2. A cone rests on a rough horizontal plane and a cord, fastened to the vertex of the cone, passes over a smooth pulley at the same height as the top of the cone and supports a load. Show that, if the load is slowly increased, the cone will turn over or slide according as the coefficient of friction μ is greater or less than $\tan\alpha$, where α is the semi-vertical angle of the cone.

[*U. P. E. S.*, 1963]

Let W be the weight of the cone and P the force applied to the vertex A in the horizontal direction due to the load.



Suppose the frictional force F of the plane on the cone is sufficient to prevent sliding, and the cone is about to turn over about the base point C, then the normal reaction R will act at this point C.

Resolving the forces horizontally and vertically, we get

$$P=F \text{ and } W=R. \qquad \ldots \quad (1)$$

Taking moments about the point C :

$$Wr-Ph=0,$$

or $$P=W.\frac{r}{h}=W\tan\alpha,$$

or, by (1), $$F=R\tan\alpha.$$

Therefore, if $\mu>\tan\alpha$, $F<\mu R$ and the cone will turn over, before the case of limiting friction arises.

If $\mu<\tan\alpha$, then the force F required to prevent sliding is greater than μR. As this is not possible, the cone will slide before it turns over.

Ex. 3. A heavy uniform rod of length $2a$ lies over a rough peg with one end leaning against a rough vertical wall. If c be the distance of the peg from the wall and λ the angle of friction both at the peg and the wall, show that when the point of contact of

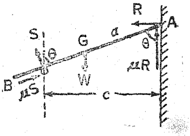
the rod with the wall is above the peg, then the rod is on the point of sliding downward when

$$a \sin^3 \theta = c \cos^2 \lambda,$$

where θ is the inclination of the rod with the wall.

[*Jabalpur*, 1959]

When the contact point A of the rod with the wall is about to slip downward, then the limiting frictional force μR of the wall and μS of the peg act on the rod in the directions indicated in the figure.



The free-body diagram for the rod AB gives forces W, S, μS, R and μR in equilibrium.

Resolving the forces horizontally and vertically:

$$\mu S \sin \theta = R + S \cos \theta. \quad \ldots \quad (1)$$

$$\mu R + S \sin \theta + \mu S \cos \theta = W. \quad \ldots \quad (2)$$

From (1) and (2), eliminating R,

$$S \sin \theta \, (\mu^2 + 1) = W,$$

or

$$S \sin \theta \sec^2 \lambda = W. \quad \ldots \quad (3)$$

Taking moments about the point A:

$$W \, . \, a \sin \theta - S \, . \, c \operatorname{cosec} \theta = 0,$$

or

$$Sc = Wa \sin^2 \theta. \quad \ldots \quad (4)$$

Eliminating S from (3) and (4), we get

$$a \sin^3 \theta = c \cos^2 \lambda.$$

Ex. 4. A solid homogeneous hemisphere rests on a rough horizontal plane and against a smooth vertical wall. Show that if the coefficient of friction be greater than $\frac{3}{8}$, the hemisphere can rest in any position, and if it be less, the least angle that the base of the hemisphere can make with the vertical is

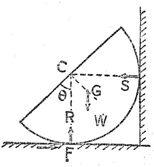
$$\cos^{-1}(\tfrac{8}{3}\mu),$$

where μ is the coefficient of friction between the plane and the hemisphere.

If the wall be rough (coefficient of friction μ_1), show that this angle is

$$\cos^{-1}\left(\frac{8\mu}{3} \cdot \frac{1+\mu_1}{1+\mu\mu_1}\right).$$

Let θ be the angle that the base of the hemisphere makes with the vertical. The forces acting on the hemisphere are the weight W, the normal reaction R of the ground, the frictional force F, and the normal reaction S of the wall. Resolving the forces horizontally and vertically, we get



$$F=S \quad \text{and} \quad R=W.$$

Taking moments about the centre C of the hemisphere,

$$Fa-\tfrac{3}{8}Wa\cos\theta=0.$$

Therefore
$$\frac{F}{R}=\frac{3}{8}\cos\theta. \qquad \dots \quad (1)$$

If $\mu>\frac{3}{8}$, then F/R is less than μ for all values of θ. The limiting friction is never reached and the hemisphere can rest in any position.

If $\mu<\frac{3}{8}$ then for the limiting case $F/R=\mu=\frac{3}{8}\cos\theta$, by (1). Therefore

$$\theta=\cos^{-1}\left(\frac{8\mu}{3}\right).$$

If the wall be rough and θ be such that the hemisphere is about to slip, then in addition to the above forces there is a frictional force $\mu_1 S$ of the wall acting vertically upwards. Resolving the forces, we get

$$\mu R=S \text{ and } R+\mu_1 S=W,$$

or
$$R(1+\mu\mu_1)=W. \qquad \dots \quad (2)$$

Taking moments about C,

$$(\mu R+\mu_1 S)a-\tfrac{3}{8}aW\cos\theta=0,$$

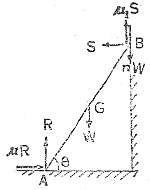
or
$$\mu R(1+\mu_1)=\tfrac{3}{8}W\cos\theta. \qquad \dots \quad (3)$$

From (2) and (3),

$$\theta=\cos^{-1}\left(\frac{8\mu}{3}\cdot\frac{1+\mu_1}{1+\mu\mu_1}\right).$$

Ex. 5. A ladder on a horizontal floor leans against a vertical wall. The coefficient of friction of the floor and the wall with the ladder are μ and μ_1 respectively. If a man, whose weight is n times that of the ladder, wants to climb up the ladder, show that the minimum safe angle of the ladder with the floor is

$$\tan^{-1}\left\{\frac{2n+1-\mu\mu_1}{2\mu(1+n)}\right\}. \qquad [\textit{Jodhpur}, 1965]$$



Let the length of the ladder be $2a$ and its weight, acting at the mid-point G, be W.

The largest tendency of the ladder slipping will be when the man is at the top of it. Let θ be the angle the ladder makes with the floor, when it is about to slip. Then the frictional forces acting at the ends A and B of the ladder are μR and $\mu_1 S$ in the directions shown in the figure. R and S are the normal reactions of the floor and the wall on the ladder. The free-body diagram for the ladder is as shown.

Resolving the forces horizontally and vertically, we get

$$S=\mu R.$$

$$R+\mu_1 S=(n+1)W.$$

Eliminating S, we have

$$R(1+\mu\mu_1)=(n+1)W. \quad \ldots \quad (1)$$

Taking moments about the point B, we get

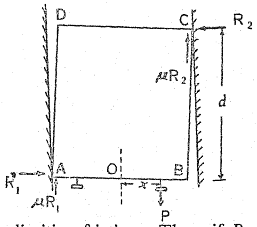
$$W \cdot a\cos\theta+\mu R \cdot 2a\sin\theta-R \cdot 2a\cos\theta=0,$$

or
$$2\mu R\sin\theta=(2R-W)\cos\theta. \quad \ldots \quad (2)$$

From (1) and (2), $2\mu\sin\theta=\left(2-\dfrac{1+\mu\mu_1}{n+1}\right)\cos\theta,$

or
$$\theta=\tan^{-1}\left\{\frac{2n+1-\mu\mu_1}{2\mu(n+1)}\right\}.$$

Ex. 6. A drawer of a table, d feet deep from back to front, has two handles. Show that if it is required to open or to shut the drawer by one handle, then the handle should not be placed at a distance greater than $\frac{1}{2}d\cot\lambda$ from the centre of the drawer, where λ is the angle of friction. [*U. P. E. S.*, 1964]



In the margin is a sectional plan of the table and the drawer $ABCD$. Suppose it has two handles at distances x feet from the centre O of the drawer.

Suppose a force P is applied to one of the handles. Due to this eccentric force P, the two corner edges A and C of the drawer will rub against the walls of the table as shown, and frictional forces opposing motion will develope there. Suppose the friction is the limiting friction. Then if R_1

and R_2 are the normal reactions of the walls at A and C, the frictional forces at A and C are μR_1 and μR_2.

Resolving the forces along AB, we get $R_1=R_2$; and taking moments about O, we get

$$\mu R_1 \,.\, OA-\mu R_2 \,.\, OB-R_2 \,.\, d+P \,.\, x=0,$$

or
$$R_2 d=Px.$$

Now, the maximum force which can be called into play to oppose the motion of the drawer

$$=\mu R_1+\mu R_2=2\mu R_2=2\mu Px/d.$$

This must not be greater than P otherwise the drawer will not move, i.e.,

$$2\mu Px/d<P,$$

or
$$x<d/2\mu, \quad \text{or} \quad x<\tfrac{1}{2}d\cot\lambda.$$

EXAMPLES 7

1. A body, weighing 5 pounds, is placed on a rough horizontal table. The body just moves when a string attached to it is pulled with a force of 1 pound at an angle inclined upwards at 30° to the table. Find the coefficient of friction.
[*Banaras*, 1954]

2. A rough rectangular block of weight W_1 is placed on another rough block of weight W_2, both being placed on a rough horizontal table. A horizontal force P acts on the upper block. Draw free-body diagrams for each of the blocks when $P<\mu W_1$, and also for the case when $P>\mu W_1$; where μ is the coefficient of friction for all the surfaces.

3. A uniform cube, whose edges are each two metres long, stands on a rough horizontal plane. A gradually increasing horizontal force is applied to one of its vertical faces at a height of 50 cm. above the centre of that face. Determine how the equilibrium will be broken (*a*) when the coefficient of friction $\mu=0{\cdot}5$, (*b*) when the coefficient of friction $\mu=0{\cdot}7$.

4. A uniform ladder rests in limiting equilibrium with one end on a rough floor whose coefficient of friction is μ, and with the other end against a smooth vertical wall. Show that its inclination with the vertical is

$$\tan^{-1}(2\mu).$$
[*Aligarh*, 1965]

5. A uniform ladder rests in limiting equilibrium with one end on a rough wall and the other end on rough ground. If the

coefficients of friction be μ_1 and μ_2 for the ground and the wall respectively, show that

$$\tan\theta=\frac{1-\mu_1\mu_2}{2\mu_1},$$

where θ is the angle of inclination of the ladder with the ground.

6. A uniform ladder, 70 ft. long, is equally inclined to the vertical wall and to the horizontal ground, both of which are rough. The weight of the man and his burden, ascending the ladder is 2 cwt., and the weight of the ladder is 4 cwt. If the coefficient of friction between the ladder and the ground is $\frac{1}{3}$ and $\frac{1}{2}$ between ladder and the wall, prove that the man may ascend 20 feet along the ladder before it begins to slip. [*Panjab*, 1956]

7. A 25-kg. ladder, 4 m. long, rests on a rough floor and against a rough wall. The ladder is inclined at 30° with the vertical and starts to slip when an 80-kg. man has climbed half-way up. If the coefficient of friction at the wall is 0·2, find the coefficient of friction at the floor.

8. A ladder, 20 ft. long with its centre of gravity 8 ft. up, weighs 60 pounds and rests at an angle θ to the ground against a smooth vertical wall. The coefficient of friction between the ladder and the ground is $\frac{1}{2}$. Find the least value of θ which will enable a man weighing 140 pounds to reach the top without the ladder slipping. [*Banaras*, 1953]

9. A uniform ladder rests at an angle of 45° with the horizontal, with its upper extremity against a rough vertical wall and its lower extremity on the ground. If μ and μ' be the coefficients of friction between the ladder and the ground, and between the ladder and the wall respectively, show that the least horizontal force that will move the lower extremity towards the wall is

$$\tfrac{1}{2}W.\frac{1+2\mu-\mu\mu'}{1-\mu'},$$

where W is the weight of the ladder.

10. A uniform rod of length $2a$ rests on a rough floor and against the smooth edge of a table of height h. If the rod is on the point of slipping when inclined at an angle θ with the horizontal floor, show that

$$\frac{1}{\mu}=\frac{h}{a}\operatorname{cosec}^2\theta\sec\theta-\cot\theta,$$

where μ is the coefficient of friction between the floor and the rod.

11. A chest in the form of a rectangular parallelopiped, whose weight without the lid is 100 kg., and width from back to front 30 cm., has a lid weighing 25 kg. and stands with its back 15 cm.

from a smooth wall and parallel to it. If the lid be open and lean against the wall, find the least coefficient of friction between the chest and the ground so that there may be no motion.

12. A pair of compasses describes a circle of radius r on a rough piece of paper. One leg of the compass is at the centre of the circle and the other leg, carrying a pencil, describes the circle. If the vertical pressure given at the head of the compass be P and the coefficient of friction between the pencil and the paper be μ, show that the horizontal couple exerted at the head is

$$\tfrac{1}{2}\mu Pr.$$

13. Two equal and uniform rods AB and BC, each of length l, are smoothly hinged at B. Two light rings are attached to the ends A and C of the rods, which can slide on a rough horizontal rod AC. If μ be the coefficient of friction between the rings and the rod AC, show that the maximum distance at which the rings can remain in equilibrium is

$$\frac{4\mu l}{\sqrt{(4\mu^2+1)}}.$$

14. Two equal uniform ladders hinged together at one end, stand with the other ends on a rough horizontal plane. A man whose weight is equal to that of one of the ladders ascends one of them. Find which of the two ladders will slip first. If the slipping occurs when he has ascended a distance x, prove that the coefficient of friction is $(a+x)\tan\alpha/(2a+x)$, a being the length of each ladder, and α the angle each makes with the vertical.

15. Two ladders, AB and AC of lengths 4 m. each and each weighing 30 kg., are smoothly hinged at A and are placed symmetrically with their ends B and C on a rough horizontal floor. The coefficient of friction between the ladder and the floor is $\frac{1}{2}$. If a man, weighing 70 kg., wants to stand anywhere on the ladder, find the maximum distance at which the ends B and C can be placed, without causing the ladder to slip.

16. A uniform rod AB, of weight W, rests with one end A against a rough vertical wall and the other end B is supported by a string equal in length to the rod and fastened to a point C vertically above A. If the rod be in limiting equilibrium, show that the inclination of the rod with the vertical is

$$\theta=\tan^{-1}(3/\mu). \qquad \textit{[Roorkee, 1955]}$$

17. A uniform rod of length $2a$, lies on a smooth peg with its end leaning against a rough wall, the point of contact below the level of the peg. If the horizontal distance of the peg from the wall

be c and λ be the angle of friction between the rod and the wall, show that the inclination θ of the rod with the vertical is given by

$$a \sin^2 \theta \sin (\theta - \lambda) = c \cos \lambda,$$

when the rod is on the point of slipping upwards, and by

$$a \sin^2 \theta \sin (\theta + \lambda) = c \cos \lambda,$$

when it is on the point of slipping downwards.

18. A rectangular panel of weight W and height h, is guided vertically and is just being raised by a vertical force P acting in a line distant e to the right of the vertical centre line of the panel. Show that

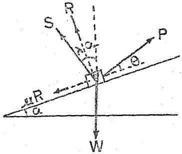
$$P = \frac{W}{1 - 2\mu e/h},$$

where μ is the coefficient of friction between the panel and the guide. Prove also that when $e \geqslant h/2\mu$, the panel gets jammed.

6·4. Inclined plane. We have already seen that a weight resting on a rough inclined plane will be in limiting equilibrium when the inclination is equal to the angle of friction. We shall consider below some further cases of the inclined plane.

Case I. *To find the force required to move a body up an inclined plane.*

Let W be the weight of a body placed on a plane inclined at an angle α to the horizontal and let P be a force applied to the body at an angle θ with the plane just inducing the body to move up the plane.



Replace R and μR by the resultant reaction S, which makes an angle λ with the normal to the plane. Then only three forces act on the body and, by Lami's theorem, we have

$$\frac{P}{\sin\{180 - (\lambda + \alpha)\}} = \frac{W}{\sin (90 - \theta + \lambda)},$$

or $$P = \frac{W \sin(\lambda + \alpha)}{\cos(\theta - \lambda)}. \quad . \quad . \quad . \quad (1)$$

(i) For a minimum P we must have $\lambda = \theta$, and

$$P_{\min} = W \sin(\lambda + \alpha).$$

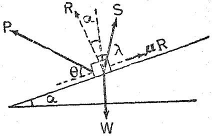
(ii) If the force P acts along the plane, $\theta = 0$, and

$$P = W \sin(\lambda + \alpha)/\cos\lambda.$$

(iii) If the force P acts horizontally, $\theta = -\alpha$, and

$$P = W \tan(\lambda + \alpha).$$

Case II. *To find the force required to move a body down an inclined plane.*



Let the inclination α be less than the angle of friction λ, and let the force P be applied to the body at an angle θ with the downward direction of the plane, just inducing the body to move down the p ane.

Replace R and μR by the resultant reaction S; then, by Lami's theorem, we have

$$\frac{P}{\sin\{180 - (\lambda - \alpha)\}} = \frac{W}{\sin(90 - \theta + \lambda)},$$

or $$P = \frac{W \sin(\lambda - \alpha)}{\cos(\theta - \lambda)}. \quad . \quad . \quad . \quad (2)$$

(i) For a minimum P, $\theta = \lambda$, and

$$P_{\min} = W \sin(\lambda - \alpha).$$

(ii) If P acts along the plane, $\theta = 0$, and

$$P = W \sin(\lambda - \alpha)/\cos\lambda.$$

(iii) If P acts horizontally, $\theta = \alpha$, and

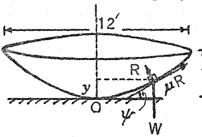
$$P = W \tan(\lambda - \alpha).$$

NOTE. If $\alpha > \lambda$, then the body will move down the plane, unless supported by a force. To find the

force P which will just support it, we can reverse the direction of P in the figure on page 92 and proceed as above. This only means that the sign of P will be reversed in the above equations.

Ex. 1. How high can a body rest inside a fixed parabolic bowl which has 12 ft. diameter at the top and is 3 ft. deep, if the coefficient of static friction is 0·6 ?

Suppose the body is in limiting equilibrium at height y ft. from the bottom of the bowl. Then the forces acting on the body are the normal reaction R, the limiting friction μR, and the weight W. Resolving along and perpendicular to the tangent, we get



$\mu R = W \sin \psi$ and $R = W \cos \psi$.

Therefore $\tan \psi = \mu = 0{\cdot}6.$ . . . (1)

The surface of the bowl is a parabola of revolution, whose equation, with axes chosen as in the figure is $x^2 = ky$. Since (6, 3) is one of the points on it, therefore

$$36 = k \times 3 \quad \text{or} \quad k = 12.$$

Hence the equation is $x^2 = 12y$, so that

$$\tan \psi = \frac{dy}{dx} = 2x/12 = x/6.$$

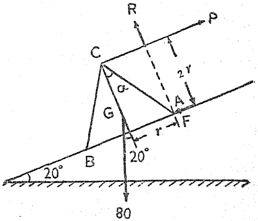
Comparing with (1) we get

$$x = 3{\cdot}6 \quad \text{and} \quad y = 3{\cdot}6 \times 3{\cdot}6/12 = 1{\cdot}08 \text{ ft.}$$

Hence the maximum height at which the body can rest is 1·08 ft.

Ex. 2. A homogeneous cone whose height is equal to the diameter of its base, is placed on a plane inclined at 20° with the horizontal. The coefficient of static friction between the base and the plane is 0·4. A force, parallel to the incline and directed up the plane, is applied to the vertex. This force is gradually increased until motion begins. Will the cone slide or tip over ? Calculate the magnitude of the applied force and of the frictional force when the motion is about to ensue. The weight of the cone is 80 kg.

Suppose the frictional force is sufficient and the cone instead of sliding tips over, under the action of the force P as shown in the figure. Under this circumstance the normal reaction R would be shifted to its outermost edge A. Let F be the force of friction. The fourth force is the weight 80 kg. of the cone acting through the centre of gravity G.



Taking moments about the point A :

$$P \cdot 2r = 80 \cos 20° \cdot r + 80 \sin 20° \cdot \tfrac{1}{2}r$$

or $\quad P = 40 \cos 20° + 20 \sin 20° = 40 \times \cdot 9397 + 20 \times \cdot 342 = 44 \cdot 43.$

Resolving the forces parallel, and perpendicular to the plane :

$$P = F + 80 \sin 20°, \qquad \ldots \quad (1)$$

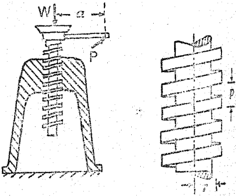
$$R = 80 \cos 20°.$$

From (1), $\quad F = 44 \cdot 43 - 80 \times \cdot 342 = 17 \cdot 07$ kg.

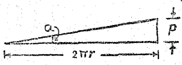
$$R = 80 \times \cdot 9397 = 75 \cdot 18 \text{ kg.}$$

The value of the limiting friction $= \mu R = \cdot 4 \times 75 \cdot 18 = 30 \cdot 07$ kg. Since the value of F obtained is less than the limiting friction hence the cone tips over before sliding can occur.

6·5. The Screw. The application of the inclined plane is found in the screws of various machines, specially in a screw-jack. A screw-jack is a machine used to lift heavy articles by small efforts. The load is placed on the top of a screw which can move up or down in a fixed nut. The effort is applied to the screw by means of a lever arm.



The thread of a screw (see figure) is an inclined plane, wound around a cylinder. The distance between two consecutive threads is called the pitch of the



screw. In the case of a single threaded screw, the inclination α of the plane is given by:

$$\tan \alpha = \frac{\text{pitch of the screw}}{\text{mean circumferential length}}$$

$$= \frac{p}{2\pi r}.$$

When a screw is used to lift a load, the weight W of the load acts vertically down and is in effect pushed up the inclined plane by a horizontal force, caused by the effort P at the lever end.

If a is the length of the lever arm, and r the mean radius of the screw, then taking moments about the axis of the screw, we see that the effective horizontal force F pushing the load up the incline is given by

$$Pa = Fr.$$

By case I (iii) of § 6·4, the force required to raise a weight W up an incline α is

$$F = W \tan(\lambda + \alpha).$$

Therefore, the effort required to raise a load W by the screw-jack is

$$P = Fr/a = (rW/a) \tan(\lambda + \alpha).$$

Similarly, by case II (iii) of § 6·4, the effort required to lower a load W is

$$(rW/a) \tan(\lambda - \alpha).$$

Ex. A screw-jack has a square-threaded screw of 3 inches mean diameter. The angle of inclination of the threads is 3° and the coefficient of friction is 0·06. It is operated by a handle 18 inches long. What pull must be exerted at the end of this handle (*a*) to raise, (*b*) to lower a load of 2 tons ? [*Panjab*, 1958]

Here $\alpha = 3°$ and $\lambda = \tan^{-1}(0{\cdot}06) = 3° 26'$.

Therefore the effort required to raise 2 tons

$$= (2r/a) \tan(\lambda + \alpha) \text{ tons}$$

$$= (2 \times 1{\cdot}5/18)(\tan 6° 26') \text{ tons}$$

$$= \tfrac{1}{6} \times 0{\cdot}1128 \times 2240 \text{ lb.} = 42{\cdot}1 \text{ lb.}$$

The effort required to lower 2 tons

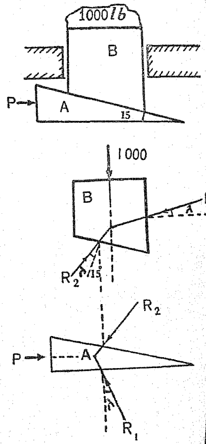
$=(2r/a)\tan(\lambda-\alpha)$ tons

$=(2\times1{\cdot}5/18)(\tan 26')$ tons

$=\frac{1}{6}\times0{\cdot}0076\times2240$ lb. $=2{\cdot}8$ lb.

6·6. Wedges. Wedges are used in engineering practice as cotter-pins, keys, etc. in which friction is a desirable element and are utilised for connecting bodies together; or they are used for raising loads.

For the solution of problems on wedges, free-body diagrams for the various connecting parts are considered separately. The resultant reaction acting on one part due to another is equal and opposite to the resultant reaction acting on the second part due to the first. The free-body diagram for any part generally gives 3 forces in equilibrium; and an application of Lami's theorem gives the solution.



Graphical methods can also be readily applied.

Ex. Find the value of the force P which must be applied to the wedge A in order to raise the block B, carrying a load of 1000 lb., assuming that μ for all surfaces of contact is 0·3. Assume that the block is in contact with only one of the vertical faces.

Here $\tan\lambda=\cdot3$, and $\lambda=16^\circ\,42'$

Free-body diagrams for the wedge A and the block B are drawn, in the figure. As the force P tends to move the wedge A to the right the frictional force acting on it will be to the left. For the convenience of solution the resultant reaction acting on each face is taken. Then by Lami's theorem, for the wedge A:

$$\frac{P}{\sin(180^\circ-2\lambda-15^\circ)}=\frac{R_2}{\sin(90^\circ+\lambda)}=\frac{R_1}{\sin(90^\circ+\lambda+15^\circ)},$$

or

$$\frac{P}{\sin(2\lambda+15^\circ)}=\frac{R_2}{\cos\lambda}=\frac{R}{\cos(\lambda+15^\circ)}. \quad \ldots$$

Similarly, for thebody B :

$$\frac{1000}{\sin(90^\circ+2\lambda+15^\circ)}=\frac{R_2}{\sin(90^\circ-\lambda)}=\frac{R_3}{\sin(180^\circ-15^\circ-\lambda)},$$

or
$$\frac{1000}{\cos(2\lambda+15^\circ)}=\frac{R_2}{\cos\lambda}=\frac{R_3}{\sin(15^\circ+\lambda)}. \quad . \quad . \quad . \quad (2)$$

From (1) and (2), we get

$$\frac{P}{\sin(2\lambda+15^\circ)}=\frac{R_2}{\cos\lambda}=\frac{1000}{\cos(2\lambda+15^\circ)}.$$

$$\therefore \quad P=1000\tan(2\lambda+15^\circ)=1000\tan 48^\circ 24'$$
$$=1263 \text{ lb.}$$

EXAMPLES 8

1. An inclined force of 150 kg. wt. is the least force which will drag a body of 300 kg. along a certain horizontal plane. Find the least force necessary to start the body up an equally rough plane, inclined at 30° to the horizontal.

2. A weight of 60 kg. is on the point of sliding down a rough inclined plane when supported by a force of 24 kg. wt. acting parallel to the plane; and it is on the point of sliding up the plane when pulled by a force of 36 kg. wt. parallel to the plane; find the coefficient of friction between the body and the plane.

3. A body, weighing 100 lb., rests on a rough plane inclined at an angle of 30° with the horizontal, and the body is held in position by a rope making an angle of 30° with the inclined plane. What are the greatest and the least tensions in the rope if the coefficient of friction is 0·25 ? [*Banaras*, 1953]

4. A right circular cylinder, 15 cm. in diameter and 45 cm. high, rests with its base on a rough table, the coefficient of friction between the table and cylinder being 0·4. The table is tilted slowly until the cylinder either topples over or slides. Which will it do and what is the inclination of the table at the instant ?

5. A cone of radius r and height h rests on a rough plane and the inclination of the plane to the horizon is gradually increased. Show that the cone will slide before it topples over if the coefficient of friction is less than $4r/h$. [*Jodhpur*, 1965]

6. A particle is at rest at the inner surface of a sphere of radius r. If the coefficient of friction be μ, show that the greatest distance of the particle from the vertical diameter is

$$\mu r/\sqrt{(1+\mu^2)}. \qquad [\textit{Roorkee}, 1966]$$

7. A cycloid is placed with its axis vertical and vertex downwards. Show that a particle can rest on it at any point which is not higher than $2a \sin^2 \lambda$ above its lowest point, where λ is the angle of friction and a is the radius of the generating circle of the cycloid.

8. If a hemisphere rests in equilibrium with its curved surface in contact with a rough plane, inclined to the horizontal at an angle α, show that the inclination of the plane base of the hemisphere is

$$\sin^{-1}\left(\tfrac{8}{3}\sin\alpha\right),$$

provided that α is less than $\sin^{-1}\frac{3}{8}$ and also less than the angle of friction. [*Banaras*, 1963]

9. A hemispherical shell rests on a rough inclined plane, whose angle of friction is λ. Show that the inclination of the plane base of the rim to the horizontal cannot be greater than

$$\sin^{-1}(2\sin\lambda).$$ [*Aligarh*, 1964]

10. The horizontal cylindrical shaft of radius r of a flywheel of weight W is turning in circular bearings with a clearance such that line contact occurs between the bearings and the shaft. Assuming dry friction with angle of friction λ between the shaft and the bearing, prove that when the wheel commences to move, the point of contact moves up from the lowest point of the bearing by a vertical distance $a(1-\cos\lambda)$, a being the radius of the bearing.

Also prove that the retarding couple on the shaft due to friction is $Wr\sin\lambda$.

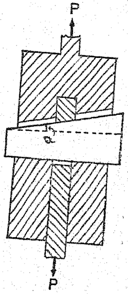
11. A flywheel is mounted on a horizontal axis which can turn in V-shaped bearings. Calculate the least weight w which will produce rotation when hung by a string round the axle of the flywheel. Assume that the weight of the flywheel and axle is 100 kg., radius of the axle is 2·5 cm., angle of V is 90° and the coefficient of friction is 0·1.

12. The axles of a four-wheeled truck of weight w are l feet apart. On the level, its centre of gravity is h feet above the road and b feet behind the front axle. Show that the maximum gradient on which the truck can remain at rest when the front wheels are locked, the rear wheels are free and the truck is facing down the gradient is

$$\tan^{-1}\frac{\mu(l-b)}{l-\mu h},$$

where μ is the coefficient of friction. [*Banaras*, 1962]

13. The centre of gravity of a motor car, standing on a level, is $2\frac{1}{2}$ feet above the road and $3\frac{1}{2}$ feet from the vertical through the back axle; the distance between the axles is $9\frac{1}{2}$ feet. When facing down a slope of $\tan^{-1}(\frac{1}{4})$ with the front wheels alone locked, it just slides down. What is μ ? Find the maximum gradient on which it can stand with back wheels locked. [*Banaras*, 1955]

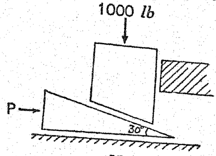


14. Prove that if the key of the cotter joint shown in the figure is not to slip when the joint resists a pull P, the value of α must not exceed $2\tan^{-1}\mu$, where μ is the coefficient of friction. If $\alpha=\tan^{-1}\mu$, find what force applied to the key in a direction perpendicular to P will be required to remove the key.

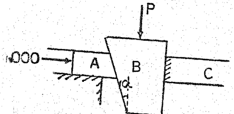
15. A square threaded screw $1\frac{1}{4}$ inch mean diameter has five threads an inch. Find the force in the direction of the axis exerted by the screw when turned against a resistance, by a handle which exerts a force equivalent to 500 lb. at the circumference of the screw, the coefficient of friction being 0·08. [*Aligarh*, 1958]

16. The mean diameter of a square threaded screw jack is 2 inches. The pitch of the thread is 0·5 inches and the coefficient of friction for the screw and the nut is 0·1. What force must be applied at the end of a lever 15 inches long to raise a load of 1 ton. [*Allahabad*, 1963]

17. In the figure the coefficients of friction for the lower wedge and the horizontal surface and the upper wedge and the vertical surface are each 0·2. The surfaces of the wedges in contact with each other are smooth. Show that the least value of the horizontal force P that will hold the 1000 lb. load is about 338 lb. Find also the least value of P which will push the lower wedge to the right. [*Punjab*, 1958]

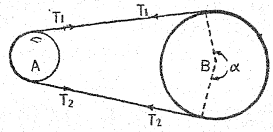


18. The given figure represents a cotter joint, the angle α of wedge $B=15°$, and the angle of friction λ of all surfaces$=12°$. What vertical force P downwards will just move the wedge B against a horizontal force of 1000 kg. at A ?



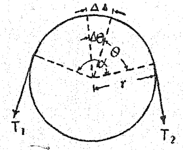
What vertical force P will just move the wedge B upwards against the same horizontal

6·7. Belt Friction. Friction of belts on pulleys is a desirable element and is utilised to transmit power. if an endless belt passes over two pulleys as shown in the figure, and if one pulley A be driven by an engine, the tensions of the belt on the two sides of the pulley become unequal because of the friction between the belt and the pulley. The resultant torque at the second pulley B causes it to rotate and the power is transmitted.

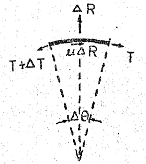


Similarly, friction of brake-bands on wheels is used to resist motion.

The portion of the-belt in which the tension is greater is called the "tight side" and the other portion in which the tension is less is called the "loose side".



(a)



(b)

The angle subtended at the centre by the portion of the belt in contact with the pulley is called the "angle of lap."

To find the relation between the tensions in the two sides of a belt (or rope) when it is about to slip over a pulley.

Let T_1 and T_2 be the tensions in the tight and loose sides of a belt and let α be the angle of lap [fig. (*a*) above].

Consider the equilibrium of an element of the belt of length $\Delta s = r\Delta\theta$. The free-body diagram for this

element showing all the forces acting on it is given in fig. (*b*).

Resolving the forces along the tangent and the normal:

$$T\cos\tfrac{1}{2}\Delta\theta+\mu\Delta R=(T+\Delta T)\cos\tfrac{1}{2}\Delta\theta,$$
$$(T+\Delta T)\sin\tfrac{1}{2}\Delta\theta+T\sin\tfrac{1}{2}\Delta\theta=\Delta R.$$

Since $\Delta\theta$ is small, up to first order terms,

$$\sin\tfrac{1}{2}\Delta\theta=\tfrac{1}{2}\Delta\theta,\quad \cos\tfrac{1}{2}\Delta\theta=1;$$

and the above equations reduce to

$$\mu\Delta R=\Delta T,\quad \text{and}\quad \Delta R=T\Delta\theta.$$

By division $$\mu=\frac{1}{T}\frac{\Delta T}{\Delta\theta}.$$

Taking limits as $\Delta\theta\to 0$, this gives

$$\frac{1}{T}\frac{dT}{d\theta}=\mu.$$

Integrating this with respect to θ between the limits 0 and α, we have

$$\int_{T_2}^{T_1}\frac{dT}{T}=\mu\int_0^\alpha d\theta,$$

or $$\log\frac{T_1}{T_2}=\mu\alpha.$$

or $$\mathbf{T}_1=\mathbf{T}_2\,\mathbf{e}^{\mu\alpha}.$$

Ex. 1. A workman lowers a heavy casting into a pit by means of a rope wrapped around a 9-in. round pole placed across the top of the pit. The coefficient of static friction for the rope on the pole is 0·4. The rope makes one complete turn around the pole. Find the greatest weight that the man can sustain by exerting a force of 50 lb. at his end of the rope. [*Banaras*, 1964]

Find also the greatest weight he can sustain, if he takes two turns of the rope around the pole.

Suppose the man can sustain a weight of W lb. Then the tension of the string on the tight side is W lb., while on the other side it is 50 lb. The angle of lap is 2π.

Therefore $$W=50e^{\cdot 4\times 2\pi},$$

or $$\log W=\log 50+\cdot 8\pi\times\log_{10} e$$
$$=1\cdot 6990+\cdot 8\times\pi\times\cdot 4343=2\cdot 7905.$$
$$\therefore\ W=617\cdot 3 \text{ lb.}$$

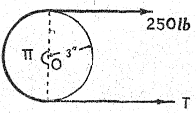
For the second part, the angle of lap is 4π (two turns); therefore

$$W=50e^{\cdot 4\times 4\pi},$$

or $$\log W=\log 50+1\cdot 6\,\pi\log e$$
$$=1\cdot 6990+1\cdot 6\times\pi\times\cdot 4343=3\cdot 8820.$$
$$\therefore\quad W=7621 \text{ lb.}$$

Ex. 2. A belt, running at slow speed, passes half round a 6″ diameter pulley. The maximum permissible tension in the belt is 250 lb. The coefficient of static friction for the belt is 0·3. Calculate the maximum torque that the belt can transmit to the pulley under these conditions. [*Allahabad*, 1962]

Let the tension on the tight side be 250 lb. (the maximum). Let tension on the loose side of the belt be T, and let the difference between tensions be such that the belt is about to slip on the pulley. Since the angle of lap is π, therefore



$$250=Te^{\cdot 3\pi},$$

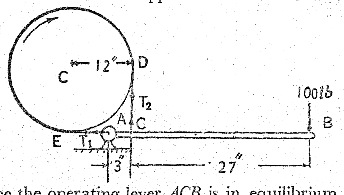
or $$\log T=\log 250-\cdot 3\times\pi\times\log e$$
$$=2\cdot 3979-\cdot 3\times\pi\times\cdot 4343=1\cdot 9889.$$
$$\therefore\quad T=97\cdot 45 \text{ lb.}$$

The maximum torque that the belt can transmit,

$$=(T_1-T_2)r$$
$$=(250-97\cdot 45)3 \text{ lb. in.}$$
$$=457\cdot 65 \text{ lb. in.} =38\cdot 14 \text{ lb. ft.}$$

Ex. 3. The figure below represents a band brake. The angle of contact is 270° and the coefficient of friction for the band and

brake wheel is 0·2. A force of 100 lb. is applied to the lever end as shown in the figure. If the brake wheel rotates in the clockwise direction, find the tensions in the band and the frictional moment developed.



Let T_1 and T_2 be the tensions in the band portions AE and DC respectively. Since the operating lever ACB is in equilibrium, hence, taking moments about A, we have

$$-100\times 30+T_2\times 3=0,$$

or $$T_2=1000 \text{ lb.}$$

Since the brake-wheel rotates in the clockwise direction, hence T_1 is the tight side and therefore

$$T_1=1000\, e^{\cdot 2\times 3\pi/2}=1000e^{\cdot 3\pi},$$

or $$\log T_1=\log 1000+\cdot 3\pi \log e$$

$$=3+\cdot 3\times 3\cdot 14\times \cdot 4343=3\cdot 4092$$

$$\therefore \quad T_1=2565 \text{ lb.}$$

The frictional moment $=(T_1-T_2)r=(2565-1000)\times 1$

$$=1565 \text{ lb. ft.}$$

Examples 10

1. A body, weighing 500 kg. is raised by means of a rope which passes over a round beam, the angle of contact being 180°. If the coefficient of friction is 0·4, what is the least force which will raise the body ? What is the least force which will hold the body ?

2. A body weighing 800 kg. is held by a rope that passes over a horizontal drum, the angle of contact being 150°. If the coefficient of friction is 0·3, find the least force that will raise the body and the least force that will hold the body.

3. How many turns of a rope should be taken around a cylindrical log of wood so that a pull of 50 kg. at one end of the rope may withstand a pull of 10 metric tonnes at the other end ? The coefficient of friction between the rope and the wood is 0·35. [*Roorkee*, 1966]

4. A rope passes over a horizontal circular beam making $1\frac{1}{2}$ turns round the beam. What is the greatest weight at one end

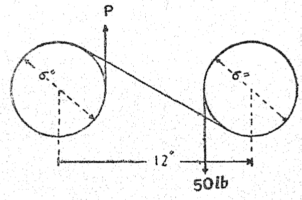
of the rope that can be supported by a force of 50 pounds, applied to the other end of the rope, if the coefficient of friction is 0·25 ? [*Allahabad*, 1965]

5. A rope is wound two times around a cylindrical post. If a pull of 40 kg. at one end of the rope will just support a force of 800 kg. at the other end, what is the coefficient of friction for the surfaces of contact ? [*Roorkee*, 1963]

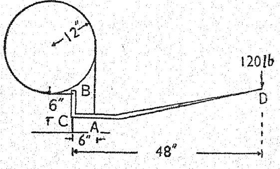
6. When a cable is wrapped once round a certain post, a pull of 40 pounds at one end withstands a load of 300 pounds at the other end. What load the same pull would withstand, if the cable were wrapped twice round the post ? [*Aligarh*, 1964]

7. A rope placed over a cylindrical drum, axis horizontal, has a weight W_1 at one end of the rope and W_2 at the other end. The system is in equilibrium for the values of W_2 ranging from 40 pounds to 250 pounds. Find W_1 and the coefficient of friction between the rope and the drum. [*Banaras*, 1962]

8. A rope is coiled round two fixed bollards as shown in the figure, and one end is held with a force of 50 lb. Find the greatest force which can be applied at the other end without causing the rope to slip. Take the coefficient of friction between the rope and the bollards to be 0·2. [*Banaras*, 1959]



9. A crude form of band brake is shown in the figure. One end of the band passing over the drum is anchored at *A* and the other at *B*. The lever can turn about *C*, and the brake is applied by pressing *D*. If the coefficient of friction between the band and the drum is 0·15, find the braking couple, and prove that it is the same whether the drum is rotating clockwise or anticlockwise.

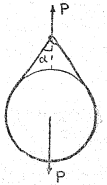


10. A rope ending in a ring is looped round a pulley wheel and subjected to a pull P in the manner shown. The coefficient of friction between the rope and ring is μ. Prove that slip between the ring and rope will occur unless α is greater than the value given by

$$2 \cos \alpha = e^{\mu\alpha},$$

and less than the value given by

$$2 \cos \alpha = e^{-\mu\alpha}.$$



CHAPTER VII

VIRTUAL WORK

7·1. Definitions. A force is said to do work when its point of application undergoes a displacement. The work done is measured by the product of the displacement and the component of the force in the direction of the displacement, when this component

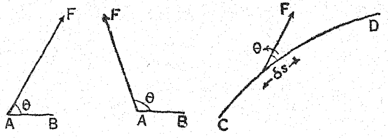


Fig. 1 Fig. 2 Fig. 3

remains constant during displacement. Thus if F be the constant force, AB the displacement of the point of its application, and θ the angle between AB and F, the work done is $AB.\ F\cos\theta$. If the angle θ be acute the work done by F is positive (fig. 1). If θ be obtuse, as in fig. 2, the work done by F is negative. We also say in a such a case that work is done against the force. If θ is 90^c, i.e. the displacement is perpendicular to the force, the work done is zero.

If the force F varies and the point of application moves along a curve CD during displacement (fig. 3), then the work done can be obtained by dividing the curve into a number of small elements and finding the sum of the work done for each element (§ 5·1, part II). The work done is

$$\int F\cos\theta\, ds,$$

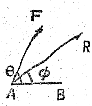
where the integral is taken from C to D along the curve.

If a system of forces, which is in equilibrium, acts upon a rigid body at rest there can be no displacement.

But it is found useful in many investigations to give the body a small fictitious displacement, and to calculate the work done by the forces for this assumed displacement. Such a displacement is called a *virtual* displacement and the work done by a force in this displacement is called *virtual work*. We always consider small virtual displacements so that the forces remain unaltered during the displacement.

7·11. Work done by resultant. *When a number of forces act on a particle which undergoes a small displacement, the algebraic sum of the work done by the various forces is equal to the work done by their resultant.*

Let A be the particle which is displaced to B. Let F be one of the forces and R the resultant, and let θ and ϕ be the angles they make with AB. Then the work done by the force F is $AB \,.\, F\cos\theta$. We shall get similar expressions for the work done by the other forces. Therefore, their algebraic sum



$$= \Sigma\, AB \,.\, F\cos\theta = AB \,.\, \Sigma F\cos\theta$$
$$= AB \,.\, R\cos\phi$$
$$= \text{work done by the resultant.}$$

7·2. Principle of Virtual Work. *If a system of forces acting on a particle be in equilibrium and the particle undergoes a small displacement, the algebraic sum of the work done by the forces is zero.* This follows immediately from the previous proposition. As the forces are in equilibrium their resultant is zero. Hence the algebraic sum of the work done by the forces, which is equal to the work done by the resultant, is also zero.

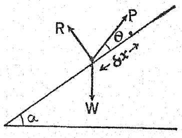
The converse of this proposition is also true; viz. *If the algebraic sum of the work done by a system of forces acting on a particle be zero for all small displacements, then the forces are in equilibrium.* For, the algebraic sum of the work done is equal to the work done by the resultant. If this is zero, either the resultant is zero or it is at right angles to the displacement. Since the sum of the work is zero for *all* small displacements, the resultant must be zero, so that the forces are in equilibrium.

These propositions are known as the *Principle of Virtual Work* for forces acting on a particle. We shall now prove this principle for coplanar forces acting on a rigid body.

NOTE. Since the forces may alter during displacement by small quantities of the order of the displacement, therefore, in general, the virtual work is not exactly zero but of an order of smallness higher than the first. However, we shall disregard quantities of order higher than the first in the discussions which follow and say that the virtual work is zero.

Ex. A particle of weight W is placed on a smooth plane inclined at an angle α to the horizontal. Determine the force P, acting at an angle θ to the plane, which will just support the particle.

Besides W and P, the reaction R of the plane will also act on the particle in a direction perpendicular to the plane. Give a virtual displacement δx to the particle up the plane. Then the work done by the force P is



$$\delta x \,.\, P \cos \theta,$$

and the work done by W is

$$-\delta x \,.\, W \cos (90^\circ - \alpha).$$

The reaction R does no work as the displacement is at right angles to R.

Hence, by the Principle of Virtual Work,

$$\delta x \,.\, P \cos \theta - \delta x \,.\, W \sin \alpha = 0,$$

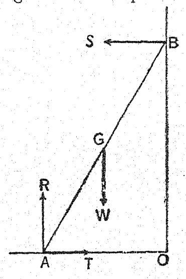
or $$P = W \sin \alpha \sec \theta.$$

7·3. Constraints. The actually possible displacements of a body or a system of connected bodies are sometimes restricted because of the manner in which the bodies are supported or connected. Thus in the example considered above physical displacement of the particle is possible only along the inclined plane. It should be noted that the reaction R of the plane does no work for such a displacement.

The geometric restrictions imposed on the displacements of a body or a system of connected bodies by the

connections or supports are called *constraints,* and the forces exerted by the connections or supports which restrict the displacements are called *constraining forces.* Thus in the example quoted above, the reaction R is the constraining force due to the constraint imposed by the inclined plane. For convenience virtual displacements are usually so chosen that they obey the constraints (i.e. the geometric restrictions imposed by the connections or supports). For such a displacement the constraining forces do no work and are sometimes called *workless* forces. The forces which do work on giving a small virtual displacement to the system are called *active* forces.

It is, however, not always necessary to give virtual displacements which obey all the constraints. In fact the latter is sometimes not even possible. For example, consider a ladder AB placed on smooth level ground OA and against a smooth vertical wall OB. Slipping is prevented by a string OA tied to the foot of the wall and the end A of the ladder. The ladder is in equilibrium under the forces indicated in the diagram. It is evident that the ground and the wall constrain the displacements of A and B to be along the ground and along the wall respectively. The string OA further restricts the displacement of A in the direction AO or OA produced.* Thus any virtual displacement of the ladder is bound to violate at least one constraint.



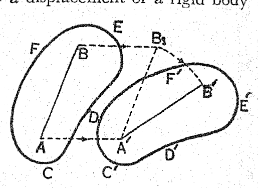
Consider, for example, a virtual displacement of the ladder in which the ends A and B slide on the ground and the wall, respectively. Then R and S are workless forces, while T and W are active forces. The tension T becomes an active force because the constraint of the string is violated. On the other hand, if a displacement is given to the ladder in which A remains fixed

*The student may think that physical displacement of A towards O is possible. However the string will become slack as soon as A moves towards O, and so the string imposes a constraint as explained earlier.

and B moves perpendicular to AB, then R and T are workless forces while S and W are active. Here S becomes an active force because the constraint of the wall is violated. It should be noted that in both the above displacements the distance AB is kept fixed, since any change in the length AB will introduce virtual work done by the internal forces acting within the ladder (these forces are not shown in the diagram).

7·4. Plane displacement of a rigid body. Consider the displacement of a rigid body in such a way that the displacements of its particles are all parallel to a given plane. This type of displacement will be produced in a body acted upon by coplanar forces.

The figure below depicts such a displacement of a rigid body from a position $CDEF$ to another position $C'D'E'F'$. Let any two points A and B in the rigid body be displaced to the positions A' and B'. Then the displacement of B is equal to $\overline{BB_1}+\overline{B_1B'}$, i.e. a displacement equal to that of A plus a rotation about A'. This is also true for the displacement for any other point.



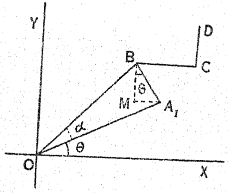
Hence if a rigid body undergoes a plane displacement, this displacement can also be achieved by giving to every particle of the body a linear displacement equal to the displacement of one of its particles A, together with a suitable rotation about A. The linear displacement given to every particle will depend on the choice of A, but the angle of rotation will be independent of this choice. Often the particle at the origin is chosen as A. Thus any plane displacement of a rigid body is equivalent to a rotation of the body about the origin, together with its translation parallel to the x-axis and a translation parallel to the y-axis.

7·5. Principle of Virtual Work for coplanar forces. *The necessary and sufficient condition that a rigid body acted by a system of coplanar forces be in equilibrium, is that the algebraic sum of the virtual work done by the forces in any small displacement of the body is zero.*

Let the coplanar forces $F_1, F_2, F_3, \ldots$ act at the points $A_1, A_2, A_3, \ldots$ of a rigid body. Let (x_1, y_1), (x_2, y_2), $(x_3, y_3), \ldots$ be the coordinates of $A_1, A_2, A_3, \ldots$ referred to a set of rectangular axes in the plane of the forces.

Let $X_1, Y_1; X_2, Y_2; X_3, Y_3; \ldots$ be the components of the forces $F_1, F_2, F_3, \ldots$ parallel to the coordinate axes.

The body can be given a displacement in the plane of the forces in three independent ways, viz. (i) a rotation about O, (ii) a translation parallel to OX, and (iii) a translation parallel to OY. Let us give all these three displacements, and suppose that the body is rotated about O through a small angle α and then given small displacements a and b parallel to OX and OY.



Let A_1, the point of application of the force F_1, be displaced by the rotation to B, and by the translation to C and finally to D. As the displacements are small, A_1B is a line of length $\alpha . OA_1$ perpendicular to OA_1. Therefore the total displacement of the point of application of F_1 parallel to OX

$$= -A_1M + BC = BC - A_1B \sin\theta$$
$$= a - \alpha . OA_1 . \sin\theta$$
$$= a - \alpha y_1.$$

Similarly, the total displacement parallel to OY

$$= MB + CD = CD + A_1B \cos\theta$$
$$= b + \alpha . OA_1 . \cos\theta$$
$$= b + \alpha x_1.$$

By § 7·11, the work done by the force F_1

= the sum of the work done by its components X_1 and Y_1

$$= (a - \alpha y_1) X_1 + (b + \alpha x_1) Y_1$$
$$= aX_1 + bY_1 + \alpha(x_1 Y_1 - y_1 X_1).$$

We shall get similar expressions for the work done by the forces $F_2, F_3, \ldots$.

Therefore, the algebraic sum of the virtual work done by the forces

$$=a\Sigma X_1+b\Sigma Y_1+\alpha\Sigma(x_1Y_1-y_1X_1). \qquad (1)$$

(i) Now if the body is in equilibrium, we have from the conditions of equilibrium

$$\Sigma X_1=\Sigma Y_1=\Sigma(x_1Y_1-y_1X_1)=0.$$

Therefore by (1), the algebraic sum of virtual work done by the forces is zero. This shows that the condition mentioned in the proposition is necessary.

(ii) Again, if the sum (1) is zero for any displacement, then

$$a\Sigma X_1+b\Sigma Y_1+\alpha\Sigma(x_1Y_1-y_1X_1)=0$$

for any arbitrarily chosen values of a, b and α which are independent of each other. This will be true only if

$$\Sigma X_1=0,\ \Sigma Y_1=0, \text{ and } \Sigma(x_1Y_1-y_1X_1)=0,$$

showing that the system is in equilibrium. This shows that the condition is sufficient to ensure equilibrium.

7·51. Omission of certain forces. In applying the Principle of Virtual Work to the solution of problems the workless forces can be omitted from consideration as the virtual work done by them is zero for a displacement consistent with the constraints of the system. Such forces are enumerated below.

(i) *The reaction of a smooth surface in contact with the body for a displacement tangential to the surface.*

As the reaction of a smooth surface is normal to the surface it is at right angles to any tangential displacement. Hence the virtual work done is zero.

(ii) *The internal reactions between parts of the body, or between two bodies of the system when the distance between them remains unaltered.*

For, the work done by the action is equal and opposite in sign to the work done by the reaction.

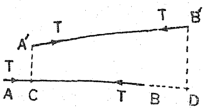
(iii) *The reaction at a fixed point or at a fixed axis of rotation.* For, the displacement of the point of application of the force is zero.

(iv) *The reaction and the frictional force when the body can roll without slipping.*

For, the point of contact being the instantaneous centre of rotation, is at rest and hence the normal reaction and the frictional force, acting at this point, have zero displacement.

(v) *The tension in an inextensible string.*

Let T be the tension in the string AB, and let a virtual displacement be given which moves the ends A and B to A' and B', respectively. Let CD be the projection of $A'B'$ on line AB. Since the string is equivalent to a force T at A and an equal and opposite force T at B, the work done by the tension



$$= T\,.\,AC - T\,.\,BD = T(AB - CD)$$

$$= T(AB - A'B' \cos\theta), \text{ where } \theta \text{ is the angle between } AB \text{ and } A'B',$$

$$= T(AB - A'B'), \text{ when } \theta \text{ is small,}$$

$$= -T\,.\,\delta l, \qquad \ldots \quad (1)$$

where δl is the extension of the string. Thus the virtual work is zero if the string is inextensible.

The relation (1) also gives the work done by the tension when the string undergoes an extension. It will be seen from the following examples that this relation is useful for obtaining the tension in a string.

(vi) *The tension or thrust in a rod whose length remains unaltered.*

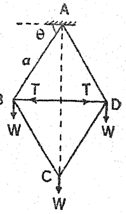
The reasons are similar to those for an inextensible string. The work done by a rod which undergoes extension can also be found similarly. The work done is $-T\,.\,\delta l$, if T is a tension, and is $T\,.\,\delta l$ if T is a thrust.

7·52. Problems on Virtual Work. The principle of virtual work gives us an alternative method to solve problems on the equilibrium of forces. A virtual displacement is given to the system and the work done by the forces is equated to zero. This equation, known as the equation of virtual work, gives a relation between the unknown quantities of the system. A suitable choice of virtual displacement will often simplify the solution of the problem. Sometimes it may be necessary to give two (or more) different displacement to obtain a sufficient number of relations between the unknown quantities.

Generally we come across two types of problems: one in which the position of equilibrium or the relation between the external forces is required, and the other in which it is required to determine the constraining forces. The example of § 7·2 in a problem of the first type, while the example of a ladder given in § 7·3 is of the second type. To solve a problem of the first type, the student should give the system a virtual displacement

obeying all the constraints. This has the advantage of removing all the constraining forces from the equation of virtual work. For an example of the second type where a particular constraining force is required a virtual displacement violating the corresponding constraint should be given. Virtual displacements of this nature are illustrated in examples 1 and 2 below.

Ex. 1. Five light rods of equal lengths are jointed together to form a rhombus $ABCD$ with diagonal BD. The frame is hung from the joint A and three equal weights, each weighing W, are suspended from the angular points B, C and D. Find the thrust in the rod BD.



Since the system is suspended from A and the weights are symmetrically placed with respect to the line AC, hence AC is vertical as shown in the figure. Let T be the thrust in the rod BD. Let AB and AD make angles θ with the horizontal and let y be the vertical depth of the points B and D below A. Then

$$\left.\begin{aligned} y&=a\sin\theta,\\ \text{and}\qquad BD&=2a\cos\theta,\end{aligned}\right\}\qquad(1)$$

where a is the length of each rod. Also, the depth of the point C below A is $2y$. Let us give the frame a virtual displacement so that θ changes to $\theta+\delta\theta$; thereby y changes to $y+\delta y$.

From principle of virtual work, we have

$$2W\,.\,\delta y+W\,.\,\delta(2y)+T\,.\,\delta(BD)=0,$$

or $$4W\,.\,\delta y+T\,.\,\delta(BD)=0,$$

or, by (1), $$4W.\ a\cos\theta\,.\,\delta\theta-T\,.\,2a\sin\theta\,.\,\delta\theta=0.$$

Therefore, $$T=2W\cot\theta.$$

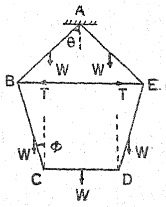
Since ABD is an equilateral triangle, therefore $\theta=60°$.

Hence $$T=2W\cot 60°=\frac{2}{\sqrt{3}}W.$$

Ex. 2. A pentagon $ABCDE$ is formed of five equal uniform rods with their extremities freely jointed together. It is suspended from the angular point A and is maintained in the form by a light rod, joining the angular points B and E. If the rods AB and AE make angles θ, and the rods BC and ED make angles ϕ with the vertical, prove that the thrust in the strut BE is

$$W(2\tan\theta+\tan\phi),$$

where W is the weight of each rod.



Since the rods are equal, hence the pentagon will hang symmetrically with respect to the vertical through A and the strut BE will be horizontal. Let T be the thrust in BE and a be the length of each rod. Let y_1, y_2 and y_3 be the vertical distances below A of the centroids of the rods AB or AE, BC or DE and CD respectively. From the principle of virtual work, we have,

$$2W\,.\,\delta y_1+2W\,.\,\delta y_2+W\,.\,\delta y_3 +T\,.\,\delta(BE)=0. \qquad (1)$$

Now

$$y_1=\tfrac{1}{2}a\cos\theta. \qquad \therefore \quad \delta y_1=-\tfrac{1}{2}a\sin\theta\,.\,\delta\theta.$$

$$y_2=a\cos\theta+\tfrac{1}{2}a\cos\phi. \qquad \therefore \quad \delta y_2=-(a\sin\theta\,.\,\delta\theta+\tfrac{1}{2}a\sin\phi\,.\,\delta\phi).$$

$$y_3=a\cos\theta+a\cos\phi. \qquad \therefore \quad \delta y_3=-(a\sin\theta\,.\,\delta\theta+a\sin\phi\,.\,\delta\phi).$$

Also $$BE=2a\sin\theta=a+2a\sin\phi.$$

$$\therefore \qquad \delta(BE)=2a\cos\theta\,.\,\delta\theta=2a\cos\phi\,.\,\delta\phi.$$

From last two equations, we obtain

$$\delta\phi=\frac{\cos\theta}{\cos\phi}\,.\,\delta\theta. \qquad . \quad . \quad . \quad (2)$$

Replacing $\delta\phi$ in terms of $\delta\theta$ from (2) in δy_2 and δy_3, we get,

$$\delta y_2=-a(\sin\theta+\tfrac{1}{2}\cos\theta\tan\phi)\delta\theta.$$

$$\delta y_3=-a(\sin\theta+\cos\theta\tan\phi)\delta\theta.$$

Putting the values of δy_1, δy_2, δy_3 and $\delta(BE)$ in the equation (1) of virtual work, we have,

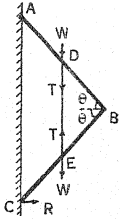
$$T\,.\,2a\cos\theta\,.\,\delta\theta=W.\,a\sin\theta\,.\,\delta\theta+2W.\,a(\sin\theta+\tfrac{1}{2}\cos\theta\tan\phi)\,\delta\theta +W\,.\,a(\sin\theta+\cos\theta\tan\phi)\delta\theta,$$

or $$T=W(2\tan\theta+\tan\phi).$$

Ex. 3. One end of a uniform rod AB, of length $2a$ and weight W, is attached by a frictionless joint to a smooth wall and the other end B is smoothly hinged to an equal rod BC. The middle points of the rods are connected by an elastic cord of natural length a and modulus of elasticity $4W$. Prove that the system can rest in equilibrium in a vertical plane with C in contact with the wall below A, and the angle between the rods is

$$2\sin^{-1}(\tfrac{3}{4}) \qquad \text{[}Banaras\text{, 1953]}$$

Let D and E be the mid-points of the rods AB and BC, to which the string is attached. Due to the weights of the rods the system will hang with C in contact with the wall and DE vertical as shown in the figure.



Let T be the tension in the string DE, and θ the angle each rod makes with the horizontal. Let y be the vertical distance of the centroid D of rod AB from A, then $3y$ is the vertical distance of the centroid E of rod BC from A. Then

$$DE=2y.$$

From the principle of virtual work, we have

$$W.\delta y+W.\delta(3y)-T.\delta(2y)=0,$$

or $$T=2W.$$

Since for elastic strings,

Force of tension = Modulus × strain,

therefore $$2W=4W.\frac{2y-a}{a},$$

or $$y=\tfrac{3}{4}a.$$

Hence $$\sin\theta=y/a=\tfrac{3}{4}.$$

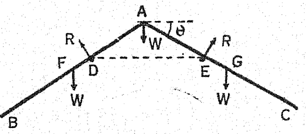
Therefore the angle between the rods $=2\theta$

$$=2\sin^{-1}(\tfrac{3}{4}).$$

Ex. 4. Two equal uniform rods AB and AC, of length a, are freely jointed at A and placed symmetrically over two smooth pegs on the same horizontal level, distance c apart. A weight, equal to that of a rod, is suspended from the joint A. Show that in the position of equilibrium the inclination of either rod with the horizontal is

$$\theta=\cos^{-1}\left(\frac{3c}{2a}\right)^{1/3},$$

provided $3c<2a$. [*Banaras*, 1961]



The adjoining diagram shows the position of the rods and the forces.

Let y_1 be the vertical depth of the centroids of the rods and y_2 be the vertical height of the weight at A, from the line of pegs DE.

Conceive a virtual displacement by raising the joint A vertically up; then from the principle of virtual work, we have,

$$2W \,.\, \delta y_1 - W \,.\, \delta y_2 = 0. \quad . \quad . \quad . \quad (1)$$

Since $\quad y_1 = GE \sin\theta = (\tfrac{1}{2}a - \tfrac{1}{2}c \sec\theta)\sin\theta = \tfrac{1}{2}(a\sin\theta - c\tan\theta)$

and $\quad y_2 = \tfrac{1}{2}c\tan\theta,$

therefore $\quad \delta y_1 = \tfrac{1}{2}(a\cos\theta \,.\, \delta\theta - c\sec^2\theta \,.\, \delta\theta),$

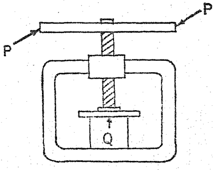
$$\delta y_2 = \tfrac{1}{2}c\sec^2\theta \,.\, \delta\theta.$$

Putting these in (1), we obtain

$$(a\cos\theta - c\sec^2\theta) - \tfrac{1}{2}c\sec^2\theta = 0,$$

whence $\quad \cos^3\theta = \dfrac{3c}{2a} \quad \text{or} \quad \theta = \cos^{-1}\left(\dfrac{3c}{2a}\right)^{1/3}.$

Ex. 5. Equal forces P are applied at the ends of the lever arms of the screw press shown in the figure. Find the compression Q exerted by the plate on the body if the pitch of the screw is h and the length of one lever arm is a.



Let us replace the compressed body by the reaction Q acting upwards. A virtual displacement compatiable with the constraints of the system will be a rotation $\delta\theta$ of the lever arms. The work done by the forces P will be $2P \,.\, a\delta\theta$. Due to this displacement the plate will move down by an amount $h \,.\, \delta\theta/2\pi$, and the work done by Q is $-Qh\delta\theta/2\pi$. Hence, by the principle of virtual work,

$$2Pa\delta\theta - Qh\delta\theta/2\pi = 0,$$

or $$Q = 4\pi aP/h.$$

Examples 10

1. A rhombus $ABCD$ is formed of four equal uniform rods freely jointed together and suspended from the point A. It is kept in position by a light rod joining the mid-points of BC and CD. Prove that if T be the thrust in this rod and W the weight of each rod of the rhombus

$$T = 4W\tan\tfrac{1}{2}A. \qquad [U.\ P.\ E.\ S.,\ 1963]$$

2. Four equal rods, each of weight W, are freely jointed so as to form a rhombus $ABCD$. The system hangs from an angular point A which is fixed and is kept in position by a weightless horizontal strut BD, attached to two angular points in the same horizontal line. A weight W' hangs from C. Show that the compression in the strut BD is

$$(2W+W') \tan \alpha,$$

where 2α is the angle of the rhombus at A. [*Banaras*, 1965]

3. Four equal uniform rods are jointed to form a rhombus $ABCD$, which is placed in a vertical plane with AC vertical and A resting on a horizontal plane. The rhombus is kept in shape, with the angle BAC equal to θ, by a light string joining B and D. Show that its tension is $2W \tan \theta$, where W is the weight of a rod. [*Aligarh*, 1965]

4. Four equal heavy uniform rods AB, BC, CD and DA, are jointed at their extremities so as to form a rhombus and the corners A and C are joined by a string. If the rhombus is suspended by the corner A, show that the tension of the string is $2W$ and that the reaction at either end B or D is

$$\tfrac{1}{2}W \tan (\tfrac{1}{2} BAD),$$

where W is the weight of each rod.

5. A string, of length a, forms the shorter diagonal of a rhombus formed of four uniform rods, each of length b and weight W, which are hinged together. If one of the rods be supported in a horizontal position, prove that the tension of the string is

$$\frac{2W(2b^2-a^2)}{b\sqrt{(4b^2-a^2)}}.$$ [*Banaras*, 1963]

6. A freely jointed framework is formed of five equal uniform rods, each of weight W. The framework is suspended from one corner, which is also joined to the middle point of the opposite side by an inextensible string. If the two upper rods make angles θ and ϕ respectively with the vertical, prove that the tension of the string is to the weight of the rod as

$$4 \tan \theta + 2 \tan \phi : \tan \theta + \tan \phi.$$ [*Banaras*, 1954]

7. Six equal uniform rods AB, BC, CD, DE, EF and FA, each of weight W, are freely jointed at their extremities so as to form a regular hexagon. The rod AB is fixed in the horizontal position and the system hangs from it and is prevented from altering its shape by means of a string, which is attached to the mid-points of the rods AB and DE. Prove that the tension in the string is $3W$.

8. A heavy elastic circular cord of natural length $2\pi a$ and weight W is placed round a smooth circular cone whose axis is vertical and semi-vertical angle is 45°. Prove that in the position of equilibrium, the tension in the cord is

$$W/2\pi,$$

and the vertical depth of the plane of the cord from the vertex of the cone is

$$a\left(1+\frac{W}{2\pi\lambda}\right),$$

where λ is the modulus of elasticity of the cord. [*Banaras*, 1947]

9. Two uniform rods AB and AC, smoothly jointed at A, are in equilibrium in a vertical plane. B and C rest on a smooth horizontal plane and the middle points of AB and AC are connected by a string. Show that the tension of the string is

$$W/(\tan B+\tan C)$$

where W is the total weight of the rods and B and C are the inclinations to the horizontal of the rods AB and AC. [*Roorkee*, 1959]

10. A uniform beam rests with its extremities on two smooth inclined planes, whose intersection is a horizontal line and whose inclinations to the horizontal are α and β $(\beta>\alpha)$. The beam is lying in a vertical plane perpendicular to the line of intersection of the planes. Show by the principle of virtual work that, in the position of equilibrium, the inclination of the beam to the horizontal is

$$\tan^{-1}\left(\frac{\cot\alpha-\cot\beta}{2}\right).$$

11. Two equal uniform rods connected by a smooth hinge are placed over the circumference of a smooth vertical circle. Show that in the position of equilibrium the inclination of either rod to the vertical is given by

$$\operatorname{cosec}^2\theta\cot\theta=l/a,$$

where $2l$ and $2a$ are the length of a rod and the diameter of the circle respectively. [*Aligarh*, 1957]

12. A rod AB is movable about a point A, and to B is attached a string whose other end is tied to a ring. The ring slides along a smooth horizontal wire passing through A. Prove by the principle of virtual work that the horizontal force necessary to keep the ring at rest is

$$\frac{W\cos\alpha\cos\beta}{2\sin(\alpha+\beta)},$$

where W is the weight of the rod and α and β are the inclinations of the rod and the string to the horizontal. [*Allahabad*, 1962]

13. Show that the force necessary to move a cylinder of radius r and weight W up a plane, inclined at an angle α to the horizontal by means of a crow-bar of length l set at an angle β to the horizontal is

$$\frac{r}{l} \cdot \frac{W \sin \alpha}{1+\cos(\alpha+\beta)}.$$

14. Six uniform heavy rods, freely hinged at their ends, form a regular hexagon $ABCDEF$ which when hung up by a point A is kept from altering its shape by two light rods BF and CE. Prove that the thrust of these rods are

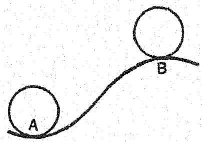
$$\frac{5\sqrt{3}}{2} W \text{ and } \frac{\sqrt{3}}{2} W,$$

where W is the weight of each rod. [*Banaras*, 1957]

15. A tripod consists of three equal uniform bars, each of length a and weight w, which are freely jointed at one extremity, their middle points being joined by strings of length b. The tripod is placed with its free ends in contact with a smooth horizontal plane and a weight W is attached to the common joint. Prove that the tension of each string is

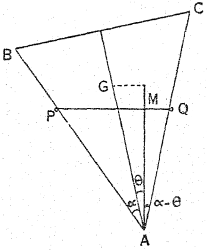
$$\frac{2(2W+3w)b}{3\sqrt{(9a^2-12b^2)}}.$$ [*Banaras*, 1955]

7·6. Stability of Equilibrium. We have so far applied the principle of virtual work to obtain positions of equilibrium. Now we shall consider the stability of these equilibrium positions. A position of equilibrium is said to be *stable* if the body slightly displaced from this position tends to return to it. A position of equilibrium is *unstable* if the body slightly displaced from that position tends to move farther away from it. In the marginal figure A is a position of stable equilibrium while the position B is unstable. In a case like a uniform sphere resting on a plane, where the body when displaced has no tendency either to return to or move farther from the former position of equilibrium, the equilibrium is called *neutral*.



If a body is in equilibrium under the force of gravity and other forces due to constraints, then for a virtual displacement obeying the constraints the forces will do no work. Therefore by the principle of virtual work, the work done by the weight will also be zero. This will be true only if the virtual displacement is horizontal (i.e., perpendicular to the weight). Hence, for equilibrium, the height of the centroid must have a stationary value. We also see from the above that for a position of stable equilibrium it must be a minimum and for unstable position it must be a maximum. As the potential energy of a body is proportional to the height of its centroid above some fixed level, we can also state this proposition as follows:

The potential energy of a body is a minimum for a position of its stable equilibrium and is a maximum for a position of unstable equilibrium.



Ex. An isosceles wedge, of height h and vertical angle 2α rests with its vertex downwards between two smooth parallel horizontal rails on the same level at distance b apart. Show that if $\frac{2}{3}h < 2b$ cosec 2α, the vertical position is stable and that unstable oblique positions of equilibrium are possible.

A section of the wedge ABC through the centroid G is shown in the figure. P and Q are the rails. Let AG make an angle θ with the vertical AM. Then from the right-angled triangles APM and AQM, we have

$$PM = AM \tan(\alpha+\theta) \text{ and } MQ = AM \tan(\alpha-\theta).$$

$$\therefore \quad PM+MQ = AM\{\tan(\alpha+\theta)+\tan(\alpha-\theta)\}$$

$$= AM\frac{\sin 2\alpha}{\cos(\alpha+\theta)\cos(\alpha-\theta)}.$$

But $PM+MQ=PQ=b$. Therefore

$$AM = \frac{b\cos(\alpha+\theta)\cos(\alpha-\theta)}{\sin 2\alpha} = \frac{b}{2\sin 2\alpha}(\cos 2\theta+\cos 2\alpha).$$

If y denotes the height of the centroid G above PQ, then

$$y = AG\cos\theta - AM = \tfrac{2}{3}h\cos\theta - \tfrac{1}{2}b\operatorname{cosec}2\alpha\,(\cos 2\theta + \cos 2\alpha),$$

so that
$$\frac{dy}{d\theta} = -\tfrac{2}{3}h\sin\theta + b\operatorname{cosec}2\alpha\sin 2\theta \qquad (1)$$

and
$$\frac{d^2y}{d\theta^2} = -\tfrac{2}{3}h\cos\theta + 2b\operatorname{cosec}2\alpha\cos 2\theta. \qquad (2)$$

From (1) we see that the positions of equilibrium are given by

$$-\tfrac{2}{3}h\sin\theta + b\operatorname{cosec}2\alpha\sin 2\theta = 0,$$

or
$$\sin\theta\,(-\tfrac{2}{3}h + 2b\operatorname{cosec}2\alpha\cos\theta) = 0,$$

i.e.
$$\theta = 0 \text{ or } \cos\theta = (\tfrac{2}{3}h/2b\operatorname{cosec}2\alpha).$$

$\theta = 0$ gives the vertical position, and the other value gives two oblique positions of equilibrium provided

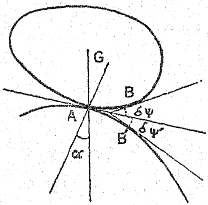
$$\tfrac{2}{3}h < 2b\operatorname{cosec}2\alpha. \quad . \quad . \quad . \quad (3)$$

For the vertical position, (2) gives

$$\left(\frac{d^2y}{d\theta^2}\right)_{\theta=0} = -\tfrac{2}{3}h + 2b\operatorname{cosec}2\alpha,$$

which is positive by (3). So y is a minimum and the vertical position is stable. As minima and maxima occur alternately the oblique positions are unstable.

7·7. Rocking Cylinders. Let the figure represent the cross sections of two cylinders whose axes are horizontal. The upper cylinder, with centre of gravity G, is free to roll on the rough surface of the lower cylinder which is fixed.



In the position of equilibrium G is vertically above the point of contact A. Consider a small displacement in which the upper curve rolls on the lower bringing the point B in contact with B'.

Let the arc $AB = AB' = \delta s$ and let the tangents at B and B' make angles $\delta\psi$ and $\delta\psi'$ with the common tangent at A. Then by calculus $\delta s/\delta\psi = \rho$ and $\delta s/\delta\psi' = \rho'$, where ρ and ρ' are the radii of curvature of the upper and lower curves at A.

Now, when the upper curve rolls on the lower bringing B in contact with B' it rotates through an angle $\delta\psi + \delta\psi'$ because the tangents at B and B' which were formerly at an angle $\delta\psi + \delta\psi'$ are now coincident. Also, by the principle of virtual work, G moves

horizontally, and as A is the instantaneous centre of rotation, the horizontal displacement of G is

$$AG(\delta\psi+\delta\psi')=h\,.\,\delta s\left(\frac{1}{\rho}+\frac{1}{\rho'}\right),\text{ where } AG=h.$$

For stable equilibrium this displacement should be less than the horizontal distance of B' from A, because then the weight acting through G will have an anticlockwise moment about the new point of contact B', and will restore the upper cylinder to its former position. If α be the angle between the common normal at A and the vertical, the horizontal distance between A and B' is $AB'\cos\alpha=\delta s\cos\alpha$. Hence the equilibrium is stable if

$$h\,.\,\delta s\left(\frac{1}{\rho}+\frac{1}{\rho'}\right)<\delta s\cos\alpha,$$

or if

$$\frac{\cos\alpha}{h}>\frac{1}{\rho}+\frac{1}{\rho'}. \qquad . \quad . \quad . \quad (1)$$

Similarly if $(\cos\alpha)/h<1/\rho+1/\rho'$, the equilibrium will be unstable.

The above considerations also apply when a spherical surface rolls on another surface.

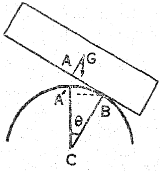
Ex. A rough uniform plank, of thickness $2b$, rests horizontally on the top of a circular cylinder of radius a. Show that the plank will be in a state of stable equilibrium if $a>b$. Supposing the condition satisfied, show that the greatest rolling displacement, for which the stability obtains, is given by

$$\tan\theta=a\theta/b.$$

Here $a=0$, $h=b$, $\rho=\infty$ and $\rho'=a$. Applying (1), we see that the equilibrium will be stable if

$$1/b>1/a, \text{ i.e. if } a>b.$$

If this condition is satisfied, let us roll the plank through an angle θ, so that the point of contact shifts from A to B. Then $AB=A'B=a\theta$. Let G be the centre of gravity of the plank. We see from the figure that stability obtains as long as the vertical through G is to the left of B. For the greatest displacement G will pass through B. In this case



$$AB\cos\theta-AG\sin\theta=0,$$

or

$$a\theta\cos\theta-b\sin\theta=0, \quad\text{or}\quad a\theta/b=\tan\theta.$$

EXAMPLES 11

1. A hemisphere rests in equilibrium on a fixed sphere of equal radius. Show that the equilibrium is unstable if the curved, and stable if the flat surface of the hemisphere rests on the sphere. [*Roorkee*, 1964]

2. A solid consists of a hemisphere and a cylinder, each of 10 inches diameter, the centre of the base of the hemisphere being at one end of the axis of the cylinder. What is the greatest length of the cylinder consistent with stability of equilibrium when the solid is resting with its curved end on a horizontal plane. [*Aligarh*, 1958]

3. A solid consisting of a cone and a hemisphere on the same base rests on a rough horizontal table with the hemisphere in contact with the table. Show that the largest height of the cone so that the equilibrium be stable is $\sqrt{3}$ times the radius of the hemisphere. [*Aligarh*, 1965]

4. A cylinder, of radius a and with axis OO' always horizontal, can roll down a perfectly rough plane inclined at an angle α to the horizontal. The cylinder is eccentrically loaded so that its centre of gravity G is distant r from OO'. Show that if $r > a \sin \alpha$, equilibrium is possible for two positions of G, and that in each case the angle which the plane $OO'G$ makes with the vertical is

$$\sin^{-1} (a \sin \alpha/r).$$

Show also that only one of these positions gives stable equilibrium.

5. A uniform rod AB rests with one end on a smooth vertical wall, and the other on a smooth inclined plane, making an angle α with the horizon. Find the positions of equilibrium and discuss stability. [*Allahabad*, 1964]

6. A solid right circular cone, of height h and semi-vertical angle α, is placed with its vertex downwards in a smooth circular hole, of radius a, cut in a horizontal table. Show that if $3h \sin 2\alpha < 16a$, there are three positions of equilibrium of which the one, with the axis of the cone vertical, is stable, and that the other two are unstable, and if $3h \sin 2\alpha > 16a$, there is only one position of equilibrium in which the axis is vertical, and this position is unstable.

7. A uniform square board, of mass M is supported in a vertical plane on two smooth pegs at the same horizontal level. The distance between the pegs is d and the diagonal of the square $D > 4d$. If one diagonal is vertical and a mass m is attached at its lower end, prove that the equilibrium is stable if

$$4md > M(D-4d).$$

8. A uniform rod AB of length $2a$ and weight w is hinged at A, a string attached to the middle point G of the rod passes over a smooth pulley at C, at a hieght a vertically above A, and supports a weight w hanging freely; find the positions of equilibrium and determine their stability. [*Roorkee*, 1965]

9. One end of a unifom rod AB of weight w and length l is smoothly hinged to a fixed point, while B is tied to a light string which passes over a small smooth pulley at a distance a vertically above A, and carries a weight $\frac{1}{4}w$. If $l<a<2l$, show that the system is in stable equilibrium when AB is vertically upwards, and that there is also a configuration of equilibrium in which the rod is inclined at a certain angle to the vertical. [*Roorkee*, 1967]

CHAPTER VIII

SUSPENDED CABLES

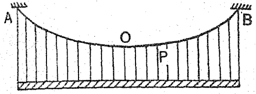
8·1. General considerations. When a rope, wire or cable is suspended from two points not in the same vertical line and carries a certain distributed load, then it assumes a curved form. The shape of the curve depends on the manner in which the load is distributed. Electric transmission wires, ropes of suspension bridges, etc. are examples of the suspension cables.

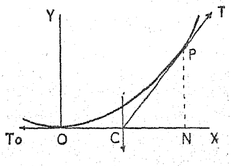
Cables, which are perfectly flexible, offer no resistance to bending. In such cases the resultant action across any section of the cable is along the tangent line to the curve formed by the cable. This is so, if we assume that the section is small and that the cable may be considered as a curved line. A chain, whose links are short and perfectly smooth, behaves like a flexible cable. These assumptions will lead to simple formulae for the shape of the cables without any serious error.

In practice two types of suspended cables are mostly found : the parabolic cable, which carries a uniformly distributed load along the horizontal, and the catenary which carries a uniformly distributed load along its length due to its own weight. These will be considered separately in the following articles.

8·2. Parabolic Cables. We shall now show that if a flexible cable suspended from two points carries a load which is uniformly distributed along its horizontal length, it takes the shape of a parabola. An example of a cable carrying such a load is the cable of a suspension bridge, in which the weight of the roadway is uniformly distributed horizontally and in which the weight of the cable and the vertical tie rods is small in comparison with the weight of the roadway. In practice the weight of the cable and the tie rods is

assumed to be distributed in the same manner as the applied load and added to the latter for design work.





Let AOB be the curve assumed by a flexible cable, which carries a uniformly distributed load w per unit horizontal length, on suspension from two points A and B.

Let O be the lowest point of the chain and P any other point of the chain whose coordinates referred to horizontal and vertical axes through O, are (x, y).

Consider the equilibrium of the portion OP of the cable. The forces acting on it are:

(i) The horizontal tension T_0 at the lowest point O;

(ii) The tension T at P along the tangent to the curve; and

(iii) The load carried by OP. As the horizontal length of OP is ON, the magnitude of the load is wx and it acts vertically through the middle point of ON.

Since these forces are in equilibrium they must meet at a point C as shown in the figure, and PNC is a triangle of forces. Therefore

$$\frac{wx}{PN}=\frac{T_0}{NC} \quad \text{or,} \quad \frac{wx}{y}=\frac{T}{\frac{1}{2}x},$$

showing that P lies on the parabola

$$x^2=(2T_0/w)y. \tag{1}$$

The tension T at P is given by

$$T=\sqrt{(T_0{}^2+w^2x^2)}. \tag{2}$$

We see that the tension in the cable increases as x increases. So the maximum tension occurs at the supports.

Quite often the two points of support are at the same level. In such cases the depth of the lowest point of the cable below the level of supports is called the *dip* or the *sag* of the cable. The horizontal distance between the two points of support is called the *span* of the cable.

The tension T_0 at the lowest point can be calculated in terms of the sag and the span of the cable. Thus, if B is the point (x_1, y_1), then sag$=y_1$ and span$=2x_1$. Therefore, by (1),

$$T_0=\frac{wx_1^2}{2y_1}=\frac{w(\text{span})^2}{8(\text{sag})}. \tag{3}$$

To find the length of the cable, we get from (1)

$$\frac{dy}{dx}=\frac{wx}{T_0},$$

so that $$\frac{ds}{dx}=\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}=\sqrt{\left\{1+\left(\frac{wx}{T_0}\right)^2\right\}},$$

and $$s_{OB}=\int_0^{x_1}\sqrt{\left\{1+\left(\frac{wx}{T_0}\right)^2\right\}}dx \tag{4}$$

$$=\tfrac{1}{2}x_1\sqrt{\left\{1+\left(\frac{wx_1}{T_0}\right)^2\right\}}+\frac{T_0}{2w}\log\left[\frac{wx_1}{T_0}+\sqrt{\left\{1+\left(\frac{wx_1}{T_0}\right)^2\right\}}\right].$$

If T_0 is large compared to wx_1, we can expand the integrand in (4) and obtain the approximate formula

$$s_{OB}=\int_0^{x_1}\left(1+\frac{w^2x^2}{2T_0^2}\right)dx=x_1+\frac{w^2x_1^3}{6T_0^2}.$$

Ex. 1. A suspension chain carries a load uniformly distributed on a horizontal platform. The load is 1·5 tonnes per metre length of the span of 200 m. and the height of the point of support above the lowest point of the chain is 15m. Find the greatest and least tensions in the chain, neglecting its weight.

If an end support (see figure of § 8·2) has the coordinates (x_1, y_1), then $x_1=100$ and $y_1=15$. Therefore, by (1) § 8·2,

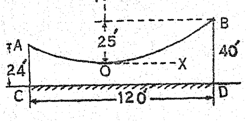
$$(100)^2=(2T_0/\tfrac{3}{2})\ 15,$$

or $$T_0=10000/20=500 \text{ tonnes}.$$

This is the least tension. The greatest tension T_1 occurs at the support and is

$$=\sqrt{(T_0^2+w^2x_1^2)}=\sqrt{(500^2+150^2)}=522{\cdot}2 \text{ tonnes.}$$

Ex. 2. A cable is hung from the top of two poles spaced 120 ft. apart on a level ground. The height of one pole is 40 ft. and that of the other 24 ft. The cable carries a uniformly distributed load of 4 lb. per foot horizontally. The cable sags 25 ft. below the top of the 40 ft. pole. Compute the tensions at the points of support and the length of the cable.



Let AOB be the cable and O its lowest point. Referred to axes through O, let the coordinates of A be (x_1, y_1), and B be (x_2, y_2). Then

$$y_1=9,\ y_2=25, \text{ and } x_1+x_2=120.$$

The equation of the cable is $x^2=(2T_0/w)y$, and since A and B lie on it, therefore

$$x_1^2=(2T_0/w)y_1=(2T_0/w)9 \text{ and } x_2^2=(2T_0/w)y_2=(2T_0/w)25. \quad (1)$$

By division, $$\frac{x_1^2}{x_2^2}=\frac{9}{25},$$

or $$\frac{x_1}{3}=\frac{x_2}{5}=\frac{x_1+x_2}{3+5}=\frac{120}{8}.$$

Therefore $$x_1=45 \text{ and } x_2=75.$$

From (1), $T_0=wx_1^2/2y_1=4\times45^2/2\times9=450$ lb.

$$T_A=\sqrt{\{T_0^2+(wx_1)^2\}}=\sqrt{(450^2+180^2)}=484{\cdot}6 \text{ lb.}$$

$$T_B=\sqrt{\{T_0^2+(wx_2)^2\}}=\sqrt{(450^2+300^2)}=540{\cdot}9 \text{ lb.}$$

Length OB of the cable

$$=\int_0^{75}\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}dx=\int_0^{75}\sqrt{\left\{1+\left(\frac{4x}{450}\right)^2\right\}}dx$$

$$=\frac{450}{4}\int_0^{2/3}\sqrt{(1+t^2)}dt, \text{ putting } \frac{4x}{450}=t,$$

$$=\frac{450}{4}\left[\tfrac{1}{2}t\sqrt{(1+t^2)}+\tfrac{1}{2}\log\{t+\sqrt{(1+t^2)}\}\right]_0^{2/3}$$

$$=80{\cdot}4 \text{ ft.}$$

Length AO of the cable

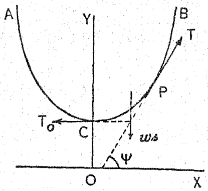
$$=\int_0^{45}\sqrt{\left\{1+\left(\frac{4x}{450}\right)^2\right\}}\,dx=\frac{450}{4}\int_0^{2/5}\sqrt{(1+t^2)}dt$$

$$=\frac{450}{4}\left[\tfrac{1}{2}t\sqrt{(1+t^2)}+\tfrac{1}{2}\log\{t+\sqrt{(1+t^2)}\}\right]_0^{2/5}$$
$$=46{\cdot}2 \text{ ft.}$$

Therefore the total length of the cable
$$=80{\cdot}4+46{\cdot}2=126{\cdot}6 \text{ ft.}$$

8·3. The Catenary. The curve a flexible cord or a chain suspended from two points assumes under the action of gravity is called a *catenary*. If the weight per unit length of the cord is constant, the catenary is called uniform or common catenary.

Let ACB be a uniform cord, of weight w per unit length, suspended from two points A and B. Let C be its lowest point, and P any other point on it. Denote the length of the arc CP by s, and the angle which the tangent to the curve at P makes with the horizontal by ψ.



Consider the equilibrium of the portion CP of the cord. The forces acting on it are :

(i) The horizontal tension T_0 at the lowest point C;

(ii) The tension T at P along the tangent to the curve; and

(iii) The weight ws of the cord CP.

Resolving these forces horizontally and vertically, we get

$$T\cos\psi=T_0, \qquad \ldots \quad (1)$$

and
$$T\sin\psi=ws. \qquad \ldots \quad (2)$$

Introduce another constant c, where

$$T_0=wc. \qquad \ldots \quad (3)$$

Then, from (1) and (2) by division,

$$s=c\tan\psi. \qquad \ldots \quad (4)$$

This is the intrinsic equation of the catenary.

To obtain its Cartesian equation, we see that

$$\frac{s}{c}=\tan\psi=\frac{dy}{dx}, \quad . \quad . \quad . \quad (5)$$

so that $$\frac{ds}{dx}=\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}=\sqrt{\left(1+\frac{s^2}{c^2}\right)}.$$

This gives on integration

$$x=\int\frac{c\,ds}{\sqrt{(c^2+s^2)}}=c\sinh^{-1}\frac{s}{c}+A.$$

If we take $x=0$ when $s=0$, then $A=0$. Therefore

$$x=c\sinh^{-1}(s/c),$$

or $$\frac{s}{c}=\sinh\frac{x}{c}. \quad . \quad . \quad . \quad (6)$$

From (5) and (6), we have

$$\frac{dy}{dx}=\sinh\frac{x}{c},$$

which gives on integration

$$y=c\cosh(x/c)+B.$$

If we take the origin at a distance c below the lowest point of the cord, the constant of integration B will be zero, and the equation of the catenary will be

$$y=c\cosh(x/c). \quad . \quad . \quad . \quad (7)$$

8·31. Definitions and Properties. The constant c occurring in the equation of the catenary is called the *parameter* of the catenary and determines its size. The lowest point C is called the *vertex* of the catenary. The horizontal line at a distance c below the vertex is called the *directrix* of the catenary. With the axes chosen above, the x-axis is the directrix.

The terms 'span' and 'sag' are used in the sense defined earlier for the catenary also.

We shall now obtain some useful relations for the common catenary. If y is the ordinate drawn to the directrix from any point, we have, from (7)

$$y^2=c^2\cosh^2\frac{x}{c}=c^2\left(1+\sinh^2\frac{x}{c}\right)=c^2\left(1+\frac{s^2}{c^2}\right),\text{ by (6)},$$

i.e. $$y^2=c^2+s^2. \quad . \quad . \quad . \quad (8)$$

Since $s=c\tan\psi$, this also gives

$$y=c\sec\psi. \qquad \ldots \quad (9)$$

The tension T at any point P is given by

$$T=\sqrt{\{T_0^2+(ws)^2\}}=w\sqrt{(c^2+s^2)}$$
$$=wy. \qquad \ldots \quad (10)$$

If the two points of suspension, A and B, are at the same level, and if the coordinates of B are (x_1, y_1), the span is $2x_1$, and the sag is y_1-c.

8·32. Approximations to the common catenary. The equation of the catenary is

$$y=c\cosh\frac{x}{c}=c\left[1+\frac{1}{2!}\left(\frac{x}{c}\right)^2+\frac{1}{4!}\left(\frac{x}{c}\right)^4+\ldots\right].$$

If x/c is small, the series on the right can be limited to the first two terms, and the equation approximates to

$$y=c+\frac{x^2}{2c}. \qquad \ldots \quad (11)$$

This shows that as long as x is small compared to c, the curve coincides very nearly with a parabola. This is so when a light cord is tightly stretched between two points, because in this case w is small while T_0 is large, and c being equal to T_0/w is also large. Examples of such a case are electric transmission wires and telegraph wires stretched between poles.

In this case, the sag

$$=y_1-c=x_1^2/2c.$$

As c is very large, the sag is small, and the tension in the cord remains very nearly equal to T_0 throughout.

The problems on tightly stretched light cords can also be tackled with the formulae of § 8·2.

Ex. 1. A telegraph line is constructed of No. 8 iron wire which weighs 7·3 lb. per 100 ft.; the distance between the poles is 150 ft. and the wire sags 1 foot in the middle. Show that it is screwed up to a tension of about 205 lb. wt. [*Allahabad*, 1964]

Since the wire is light and the sag is small therefore the curve of the wire approximates the parabola $y=c+x^2/2c$. The sag

$$=y_1-c=x_1^2/2c.$$

It is given that $y_1-c=1$ ft., and $x_1=75$ ft. Therefore

$$1=75^2/2c \text{ or } c=75^2/2.$$

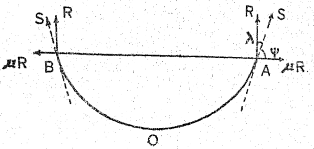
The horizontal tension

$$T_0=wc=\frac{7{\cdot}3}{100}\times\frac{75^2}{2}=205 \text{ lb. wt.}$$

Ex. 2. If the ends of a uniform inextensible string of length l hanging freely under gravity slide on a fixed rough horizontal rod whose coefficient of friction is μ, show that at most they can rest at a distance

$$\mu l \log_e \frac{1+\sqrt{(1+\mu^2)}}{\mu}, \qquad \text{[Jabalpur, 1955]}$$

Let AB be the maximum span. In this position the ends A and B are in the state of limiting equilibrium, and the frictional force developed $=\mu R$. The resultant reaction S makes an angle λ with the vertical (where $\tan\lambda=\mu$). This total reaction S must be equal and opposite to the tension of the cord at A for equilibrium, i.e., it is tangential to the curve at A. Therefore for A



$$s_1=c\tan\psi_1=c\tan(90-\lambda)=c\cot\lambda=c/\mu.$$

But $s_1=\frac{1}{2}l$, therefore $\quad \frac{1}{2}l=c/\mu, \quad$ or $\quad c=\frac{1}{2}\mu l. \qquad (1)$

To find the span we note that

$$s_1=c\sinh(x_1/c),$$

or
$$x_1=c\sinh^{-1}(s_1/c)=\tfrac{1}{2}\mu l\sinh^{-1}(1/\mu)$$
$$=\tfrac{1}{2}\mu l\log\left\{\frac{1}{\mu}+\sqrt{\left(1+\frac{1}{\mu^2}\right)}\right\}.$$

Hence the span $AB=2x_1$

$$=\tfrac{1}{2}\mu l\log\frac{1+\sqrt{(1+\mu^2)}}{\mu}.$$

Ex. 3. A cable of length 60 m. and weighing 3 kg. per metre is suspended from two points in the same horizontal plane. The tension at the point of support is 400 kg. Find the sag and the distance between the two points of support.

Let (x_1, y_1) be the coordinates of one point of support, (see figure of § 8·3). Then, by (10) § 8·32.

$$T_1=wy_1, \text{ or } 400=3y_1.$$
$$\therefore\ y_1=400/3=133{\cdot}3 \text{ m.}$$

Since $y^2=c^2+s^2$, therefore

$$(400/3)^2=c^2+(30)^2,$$

or
$$c=\sqrt{\{(400/3)^2-(30)^2\}}=129{\cdot}91 \text{ m.}$$
$$\text{Sag}=y_1-c=133{\cdot}33-129{\cdot}91=3{\cdot}42 \text{ m.}$$

Since $y_1=c\cosh(x_1/c)$, therefore

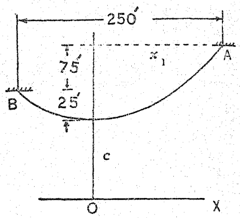
$$133{\cdot}3 = 129{\cdot}9 \cosh \frac{x_1}{129{\cdot}9} \quad \text{or} \quad \frac{133{\cdot}3}{129{\cdot}9} = \cosh \frac{x_1}{129{\cdot}9},$$

or $\cosh (x_1/129{\cdot}9) = 1{\cdot}0263 = \cosh {\cdot}2289$ (from tables).

Therefore $x_1 = 129{\cdot}9 \times {\cdot}2289 = 29{\cdot}54$ m.

Hence the span of cable $= 2x_1 = 59{\cdot}08$ m.

Ex. 4. A cable is hung from two points A and B, 250 ft. apart horizontally, and the support B is 75 ft. lower than the support A. The cable sags so that its lowest point is 25 ft. below the lower support. The cable weighs 4 lb. per foot. Find the tensions at the points of support and the length of the cable.



Let c be the parameter of the cable, then the coordinates of A are $(x_1, c+100)$, and of B are $(x_1-250, c+25)$.

The equation of the catenary is

$$y = c \cosh \frac{x}{c}, \text{ or } x = c \cosh^{-1} \frac{y}{c}.$$

Putting the coordinates of A and B in the latter equation, we have

$$\frac{x_1}{c} = \cosh^{-1}\left(1+\frac{100}{c}\right),$$

and
$$\frac{250-x_1}{c} = \cosh^{-1}\left(1+\frac{25}{c}\right).$$

Adding these, we get

$$\frac{250}{c} = \cosh^{-1}\left(\frac{100}{c}+1\right) + \cosh^{-1}\left(\frac{25}{c}+1\right).$$

or
$$z = \cosh^{-1}(1+{\cdot}4z) + \cosh^{-1}(1+{\cdot}1z),$$

where $z = 250/c$. It is now required to find the value of z which will satisfy this equation. By trial it is found that $z = 1{\cdot}65$ approximately.

$$[\cosh^{-1}(1+{\cdot}4z) + \cosh^{-1}(1+{\cdot}1z) = \cosh^{-1}(1{\cdot}66) + \cosh^{-1}(1{\cdot}165)$$
$$= 1{\cdot}09 + {\cdot}565 = 1{\cdot}655].$$

Therefore $c = 250/z = 250/1{\cdot}65 = 151{\cdot}5$ ft.

$$y_1 = c + 100 = 151{\cdot}5 + 100 = 251{\cdot}5 \text{ ft.}$$
$$y_2 = c + 25 = 151{\cdot}5 + 25 = 176{\cdot}5 \text{ ft.}$$

Since $T=wy$, therefore

$$T_A=4\times251\cdot5=1006 \text{ lb.}$$
$$T_B=4\times176\cdot5=706 \text{ lb.}$$

Since $y^2=c^2+s^2$, therefore

$$s_1=\sqrt{(y_1^2-c^2)}=\sqrt{\{(251\cdot5)^2-(151\cdot5)^2\}}=200\cdot7 \text{ ft.}$$
$$s_2=\sqrt{(y_2^2-c^2)}=\sqrt{\{(176\cdot5)^2-(151\cdot5)^2\}}=90\cdot5 \text{ ft.}$$

Hence the length of the cable $=s_1+s_2=291\cdot2$ ft.

EXAMPLES 13

1. Find the tension at the lowest point of the cable of a suspension bridge whose weight is 150 tons, evenly distributed over a span of 120 feet. The depth of the lowest point of the cable is 15 feet. [*Allahabad,* 1965]

2. In a suspension bridge of 150 metres span and 15 m. dip, the whole weight supported by the two chains is 6 tonnes per horizontal metre. Find the horizontal tension in each chain and the tensions at the points of support.

3. A suspension foot-bridge is 40 m. long and $1\frac{1}{2}$ m. wide and carries a load of 640 kg. per square metre of the floor area. It is supported by two cables which have a sag of 5 m. Find the maximum stress in each cable and the length of each cable between the supports. [*U. P. E. S.*, 1964]

4. Each cable of a four cable suspension bridge has a span of 240 feet and a sag of 30 feet. At each pier there are four stays, making 40° with the horizontal, which relieve the pier of all horizontal pull. The roadway and the bridge structure weigh 1·2 tons per horizontal foot. Find the tension in each cable at the support, the tension in each stay and the vertical load on the pier. [*Banaras,* 1954]

5. A cable is strung between two supports, one of which is 10 m. higher than the other. The sag, measured from the lower support, is 5 m. and the horizontal distance between the supports is 100 m. If the cable supports a uniformly distributed horizontal load of 60 kg. per metre, determine the tension at each support and the length of the cable. [*U. P. E. S.*, 1964]

6. A telegraph wire has a span of 30 metres and sags 15 cm. in the middle. Find the tension at the ends of the wire if it weighs $\frac{1}{4}$ kg. per metre.

7. A 100 foot steel tape is stretched tightly in measuring distances. What must be the approximate tension in the tape that the error in measurement shall not be more than 0·1 inch in each hundred feet measured? [*Banaras,* 1965]

8. A trolly wire is carried on poles round a curve of 400 metres radius. The poles are 40 metres apart and in the middle of each span the wire sags 15 cm. below the points of support. If the wire weighs 800 g. per metre find the resultant pull on each pole.

9. A uniform chain AB of length l is suspended from a fixed point A. The end B is pulled horizontally by a force equal to the weight of a length a of the chain. Find the horizontal and vertical distances between A and B. [*Roorkee*, 1954]

10. A uniform cable is 100 metres long and weighs 2·5 kg. per metre and is suspended from two points in the same horizontal plane so that the tension in the cable shall not exceed 600 kg. Find the maximum span of the cable and the corresponding sag.

11. A kite is flown with 600 feet of string from the hand to the kite and a spring balance held in the hand shows a pull equal to the weight of 100 feet of the string, inclined at 30° to the horizontal. Find the vertical height of the kite above the hand. [*Allahabad*, 1965]

12. A boat is tied by a uniform chain 20 feet long, one end of which is attached to the boat B and the other end to the top of a post A, A being 12 feet higher than B. The stream exerts a force of $7\frac{1}{2}$ lb. wt. on the boat and the chain has a mass of $\frac{1}{2}$ lb. per foot; show that the distance of B from the vertical through A is

$$30 \log_e \tfrac{5}{3} \text{ ft.}$$ [*Banaras*, 1965]

13. A heavy uniform string 90 cm. long hangs over two smooth pegs at different heights. The parts which hang vertically are of lengths 30 and 33 cm. Prove that the vertex of the catenary divides the whole string in the ratio 4 : 5, and find the distance between the pegs. [*Jabalpur*, 1959]

14. A chain of length 170 feet and total weight 255 pounds has its ends attached to two supports and hangs freely under its own weight. The tension at one support is 240 pounds and at the other 270 pounds. Find the equation of the curve in which the chain hangs and the coordinates of the points of support. [*Banaras*, 1964].

15. A uniform chain of length $2s$ is hung from two points A and B on the same horizontal line, the slope at either of these points being θ; d is the sag at the middle point. Prove that

$$\frac{\tan\theta}{s}=\frac{w}{T_0}=\frac{\sec\theta-1}{d},$$

where T_0 is the tension at the lowest point, and w the weight per unit length of chain. [*Roorkee*, 1966]

16. A telegraph wire is made of a given material, and such a length l is stretched between two posts, distant d apart and of

the same height, as will produce the least possible tension at the posts. Show that $l=(d/\lambda)\sinh\lambda$, where λ is given by the equation $\lambda\tanh\lambda=1$. [*Aligarh*, 1957]

17. A telegraph wire of length l hangs between two posts on the same level, at a distance a apart, the small sag at the centre being b. Show that $l-a=8b^2/3a$, nearly. [*Roorkee*, 1956]

18. Show that the length of an endless chain which will hang over a circular pulley of radius a so as to be in contact with two-thirds of the circumference of the pulley is

$$a\left[\frac{3}{\log(2+\sqrt{3})}+\frac{4\pi}{3}\right].$$

19. A uniform cable of weight W is suspended from two points at the same level and a weight W_1 is attached to its lowest point. If α and β are now the inclinations to the horizontal of the tangents at the highest and the lowest point, prove that

$$\frac{W}{W_1}=\frac{\tan\alpha-\tan\beta}{\tan\beta}.$$ [*Banaras*, 1953]

20. A uniform chain, of length $2l$ and weight W, is suspended from two points A and B in the same horizontal line. A load P is now hung from the middle point D of the chain and the depth of this point below AB is found to be h. Show that each terminal tension is

$$\tfrac{1}{2}\left[P\cdot\frac{l}{h}+W\cdot\frac{h^2+l^2}{2hl}\right].$$

[*Jabalpur*, 1957]

CHAPTER IX

ELASTICITY

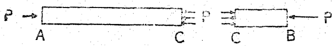
9·1. Stress and strain. So far we have been considering rigid bodies which do not change their shape or size on the application of forces. But in actual practice the bodies in nature are not rigid. They undergo a slight deformation when acted upon by forces. If the body recovers its former shape when the forces are removed, it is called an *elastic* body. If it does not recover its former shape on the removal of the forces it is in *plastic* state. Most of the solid bodies in nature are elastic to a certain extent of deformation. For larger deformations they turn into the plastic state. Steel and rubber are examples of substances which are elastic within large limits of loading, while a substance like putty is plastic from almost the initial stages.

The deformation of a body per unit of its length is known as *strain*. The simplest type of elastic deformation is a uniform extension in which each element undergoes the same strain and is extended in the same ratio. An example is that of a light steel wire fixed at one end and with a weight suspended from the other end. If l is the natural length of the wire and x the extension then x/l measures the tensile strain.

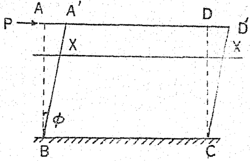
A reverse type of deformation is obtained when a rod of length l is subjected to compressive forces parallel to its length at the two end faces of the rod. If x is the compression then x/l measures the compressive strain.

When a body is acted upon by external forces, internal forces at all points within the body are called into play to resist and to balance the applied forces. If we draw a plane at a point in the body, then the magnitude per unit area of the internal forces acting on this plane is known as *stress*.

For example, if an imaginary section of a rod AB under compression is taken perpendicular to its length at C, then internal forces of total magnitude P will act at C to keep the two portions in equilibrium. If a be the area of the section at C, then the stress on this section is P/a.



Consider now a rectangular block $ABCD$ whose face BC is fixed. Let a force P, parallel to AD, be applied to the face AD. Then this force will cause the rectangular block $ABCD$ to distort to the shape $A'BCD'$. Such a deformation is known as *shear*. The deformed section $A'BCD'$ is a parallelogram and $AA' = DD'$. The shearing strain



$$= AA'/AB = \tan(ABA') = \text{angle } ABA',$$

since the strain is small.

If we divide the block by any section XX parallel to AD, the internal forces acting on this section for each part will be forces parallel to XX of total magnitude P, that on the upper part being in a direction opposite to P while that on the lower part being in the same direction as P. The magnitude of these shearing stresses are given by $P \div$ the area of the face AD.

9·2. Hooke's Law. The relation between the stress and the strain for elastic bodies was first given by Robert Hooke and is known as Hooke's law. It states that within certain limits the stress is proportional to the strain. The constant of proportionality is known as the modulus of elasticity. Thus if a bar of cross section a and length l, is subjected to a tension (or compression) P, and the extension (or compression) is x, then

$$\frac{P}{a} = E\left(\frac{x}{l}\right),$$

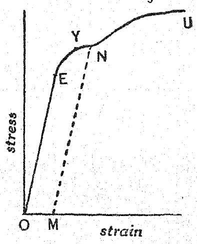
where E is called *Young's modulus* of elasticity for the material of which the bar is made.

Similarly, if the force P acting along the face AD of a block $ABCD$ produces a shearing strain ϕ, and if a is the area of the face AD, then

$$\frac{P}{a} = G\phi,$$

where G is called the *modulus of rigidity* for the material of which the block is made.

9·21. Stress-strain diagram. If a steel bar be subjected to a gradually increasing tension and the stresses and the corresponding strains be measured, a curve of the form $OEYU$ will be obtained on plotting them on a graph.



It will be found that from O up to a point E, the graph is a straight line. Hooke's law is valid in the range OE, and the stress is proportional to the strain. The stress at the point E is called the *elastic limit.*

Shortly beyond E is the *yield point* Y after which the plastic stage sets in. The portion EY is slightly curved, but beyond Y the curve is nearly horizontal and there is a considerable increase of strain for a small change in stress. If, after reaching a point N beyond the yield point the load is slowly removed, the point on the graph will trace a line NM practically parallel to OE. The residual strain OM which does not disappear on the complete removal of the load is called the *permanent set.*

After reaching a point U, the strain increases without the addition of further load and the piece begins to flow and breaks. The stress at the point U which causes the failure is called the *ultimate strength* of the material.

The greatest stress an engineering structure is likely to be subjected to is called the *working stress.* For strength and durability the working stress should be kept well within the elastic limit for the materials of the structure. The ratio of the ultimate strength to the working stress is called the *factor of safety.*

The choice of a suitable factor of safety is largely a matter of experience and judgement and depends upon the nature of the load and the reliability of the material. For instance, the factor of safety for brittle materials like cast iron is larger than that for elastic materials. It is also larger for a dynamic load than for a static one.

The values of the elastic limit, ultimate strength, and modulus of elasticity for a few materials are given below.

Material	Ultimate strength* lb./in.^{2}			Elastic limit lb./in.^{2}	uoYng's modulus lb./in.^{2}
	Tension	Compres-sion	Shear		
Steel	60,000	60,000	50,000	30,000	30×10^6
Cast iron	30,000	90,000	20,000	6,000	15×10^6
Wrought iron	50,000	40,000	40,000	25,000	28×10^6
Wood(along grain)	9,000	6,000	1,500	3,000	15×10^5

9·22. Extension of Composite Bars. If a bar of length l and cross-sectional area a suffers an alteration x in length under the action of a load P, acting normal to the cross-section, then,

$$\text{Stress}=P/a \text{ and strain}=x/l,$$

whence
$$\frac{P}{a}=E\frac{x}{l},$$

or
$$x=\frac{Pl}{aE}.$$

If the bar is a composite one, made up of two bars of different materials firmly united at the two ends, the component bars suffer the same alteration in length under the action of load P.

Let a_1 and a_2 be the cross-sectional areas of the two bars, E_1 and E_2 their Young's moduli, and P_1 and P_2 the parts of the total load carried by them respectively. Then, we have

$$\frac{P_1}{a_1}=E_1\frac{x}{l} \text{ and } \frac{P_2}{a_2}=E_2\frac{x}{l}.$$

But $P_1+P_2=P$, therefore

$$(E_1a_1+E_2a_2)\frac{x}{l}=P, \text{ or } x=\frac{Pl}{E_1a_1+E_2a_2}.$$

Ex. 1. A wrought iron bar 3 metres long is 2 cm. in diameter for 75 cm., 1·5 cm. in diameter for 1 metre and 1 cm. in diameter for the remainder of its length. This bar is in tension and the stress in the smallest section is 900 kg./cm.^{2} Find the total elongation of the bar if $E=2\times10^6$ kg./cm.^{2}

* To obtain the values in metric units (kg./cm.^{2}) divide the tabular values by 14.

Let f_1 and f_2 be the stresses in 2 cm. and 1·5 cm. diameter sections respectively and a_1, a_2, a_3 the cross-sectional areas of the three sections. Then

$$a_1 f_1 = a_2 f_2 = P = a_3 \times 900.$$

$$\therefore \qquad f_1 = \frac{a_3 \times 900}{a_1} = \frac{900 \times 1}{2^2} = 225 \text{ kg./cm.}^2,$$

and
$$f_2 = \frac{a_3 \times 900}{a_2} = \frac{900 \times 1}{(1{\cdot}5)^2} = 400 \text{ kg./cm.}^2$$

Let x_1, x_2 and x_3 be the elongations in 2 cm., 1·5 cm. and 1 cm. diameter sections; then from the formula $x = fl/E$, we get

$$x_1 = \frac{225 \times 75}{2 \times 10^6} = 0{\cdot}00844 \text{ cm.},$$

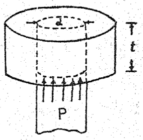
$$x_2 = \frac{400 \times 100}{2 \times 10^6} = 0{\cdot}02000 \text{ cm.},$$

$$x_3 = \frac{900 \times 125}{2 \times 10^6} = 0{\cdot}05625 \text{ cm.}$$

Total elongation $= x_1 + x_2 + x_3 = 0{\cdot}847$ mm.

Ex. 2. The cover plate of a cylinder of 12 inches inner diameter, is held by eight equally spaced wrought iron bolts near the outer edge of the cylinder head. If the maximum pressure in the cylinder is 150 pounds per square inch and if the maximum working tensile stress in bolt should not exceed 5000 pounds per square inch, show that the diameter of the bolt should be $\frac{3}{4}$ inches.

Find the shearing stress in the head of each bolt if the thickness of the head is $\frac{3}{4}$ inches.



The maximum total force on all the eight bolts = the area of the cylinder × pressure $= \pi \times 6^2 \times 150$ lb.

$\therefore$ force P on each bolt $= \frac{1}{8}(36 \times 150\pi)$
$= 675\pi$ lb.

If d be the diameter of the bolt and f_t the tensile stress in it, then

$$P = \tfrac{1}{4}\pi d^2 f_t, \text{ or } 675\pi = \tfrac{1}{4}\pi d^2 .5000,$$

whence
$$d = \sqrt{\{(675 \times 4)/5000\}} = {\cdot}735''.$$

Hence in round terms the diameter is $\frac{3}{4}$ inches.

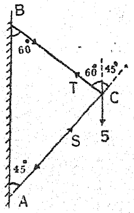
Let f_s be the shearing stress in the head of the bolt, then

$$P = \pi dt \, . f_s,$$

or
$$675\pi = \pi \times \tfrac{3}{4} \times \tfrac{3}{4} \times f_s,$$

or
$$f_s = (675 \times 16)/9 = 1200 \text{ lb./in.}^2$$

Ex. 3. In the frame, given below, the members AC and BC are of wrought iron and they make angles 45° and 60° with the vertical. A load of 5 metric tonnes is suspended from the joint C. Find the cross-sectional areas of members AC and BC taking the factor of safety as 5.



Let S be the compression in AC and T the tension in BC. The point C is in equilibrium under the action of the forces 5 tonnes, S and T. By Lami's Theorem

$$\frac{T}{\sin 45^\circ}=\frac{S}{\sin 60^\circ}=\frac{5}{\sin 105^\circ}.$$

$$\therefore \quad T=\frac{5\times\cdot 707}{\cdot 966}=3\cdot 68 \text{ tonnes,}$$

and $$S=\frac{5\times\cdot 866}{\cdot 966}=4\cdot 5 \text{ tonnes.}$$

For the member BC which is in tension the ultimate strength is $50{,}000\div 14$ kg./cm.^{2} (from the table on p. 141). Hence, the working stress for it$=\frac{1}{5}(50{,}000\div 14)=714$ kg./cm.^{2} Therefore the cross-sectional area a_1 of BC is given by

$$3\cdot 68\times 1000=a_1\times 714, \text{ or } a_1=5\cdot 15 \text{ cm.}^2$$

Similarly for the member AC in compression, the working stress $=\frac{1}{5}(40{,}000\div 14)=571$ kg./cm.^{2} and the cross-sectional area a_2 is given by

$$4\cdot 5\times 1000=a_2\times 571, \text{ or } a_2=7\cdot 88 \text{ cm.}^2$$

Ex. 4. A reinforced concrete column is 10 inches square. The principal reinforcement consists of 8 longitudinal steel rods placed along the edges of the column, each of diameter $\frac{3}{4}$ inches. The load carried by the column is 40 tons. Determine the compressive stresses in the concrete and in the steel, assuming that the moduli of elasticity for the concrete and steel are respectively 3×10^6 and 30×10^6 lb./in.^{2}

Let f_c and f_s be the compressive stresses and E_c and E_s the moduli of elasticity for the concrete and the steel respectively. Since the strain is the same for both, therefore

$$\frac{x}{l}=\frac{f_s}{E_s}=\frac{f_c}{E_c} \text{ whence } \frac{f_s}{f_c}=\frac{E_s}{E_c}=10.$$

$$\therefore \quad f_s=10 f_c.$$

The cross-sectional area of steel

$$=a_s=8\times\tfrac{1}{4}\pi d^2=2\pi\times(\tfrac{3}{4})^2=3\cdot 53 \text{ in.}^2$$

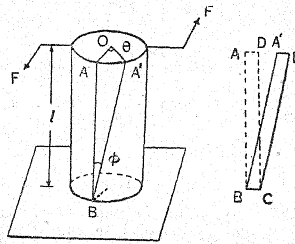
The cross-sectional area of concrete $a_c = 100 - a_s = 96{\cdot}47$ in.^{2}
Total load $= a_s f_s + a_c f_c = 40 \times 2240$,

or
$$3{\cdot}53 \times 10 f_c + 96{\cdot}47 f_c = 40 \times 2240.$$

$$\therefore \qquad f_c = \frac{40 \times 2240}{35{\cdot}3 + 96{\cdot}47} = 680 \text{ lb./in.}^2$$

$$f_s = 10 f_c = 6800 \text{ lb./in.}^2$$

9·3. Torsion. Suppose a bar of circular cross-section, fixed at one end, is twisted by a couple of moment T, applied at its free end, the axis of the couple coinciding with the axis of the bar. The effect of the couple is to give a twist to every section of the bar perpendicular to the axis. If the bar is stressed within the elastic limit, this twist increases uniformly from zero at the fixed end to an angle, $AOA' = \theta$ at the free end. A line BA, originally parallel to the axis, takes the position BA' after twisting. The angle θ through which the free end rotates due to the applied couple is called the *angle of twist*.



Because of the twist every element of the bar undergoes a shearing strain. Due to this strain shearing stresses develop on the cross-section of the bar, which balance the applied torque. We shall now calculate the moment of the shearing forces acting over a cross-section.

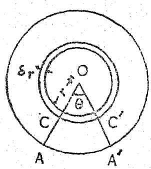
Due to the twisting, an element $ABCD$ at the surface of the bar undergoes a shearing from the position $ABCD$ to the position $A'BCD'$. The shearing strain ϕ

$$= \angle ABA' = AA'/AB = R\theta/l, \qquad (1)$$

where R is the radius and l the length of the bar.

Similarly, for a parallel element inside the bar, at a distance r from the axis, the shearing strain

$$=CC'/AB = r\theta/l.$$



The shearing stress on the cross-section of this element is

$$Gr\theta/l. \quad . \quad . \quad . \quad (2)$$

The force causing this stress is $(Gr\theta/l)\alpha$, where α is the cross-sectional area of the element. The moment of this force about the axis

$$=(Gr^2\theta/l)\alpha.$$

Therefore, if we take all the elements whose cross-sections lie on the elementary ring of radius r and width δr, the moment of the shearing forces

$$=(Gr^2\theta/l)2\pi r\,\delta r.$$

Summing up for all such rings the total moment of the shearing forces

$$=\int_0^R (Gr^2\theta/l)2\pi r\,dr = \frac{2\pi G\theta}{l}\int_0^R r^3\,dr$$

$$=\tfrac{1}{2}\pi G\theta R^4/l. \qquad (3)$$

This moment must be equal to the moment T of the applied couple. Hence

$$T=\tfrac{1}{2}\pi G\theta R^4/l. \qquad (4)$$

If f_s is the shearing force at the surface, then by (1)

$$f_s = GR\theta/l. \qquad (5)$$

We see from (2) that the maximum shearing stress on a cross-section occurs at $r=R$, and is f_s.

The quantity

$$\int_0^R r^2 . 2\pi r\,dr, \text{ i.e. } \tfrac{1}{2}\pi R^4$$

is known as the moment of inertia of the circular section (see Part II) and is denoted by I. In terms of I the relations (4) and (5) may be written as

$$\frac{T}{I}=\frac{f_s}{R}=\frac{G\theta}{l}. \qquad (6)$$

COROLLARY. If the bar is a hollow cylinder of inside radius R_1 and outside radius R_2, then instead of (3) the total moment of the shearing forces is

$$\int_{R_1}^{R_2}(Gr^2\theta/l)2\pi r\,dr=\tfrac{1}{2}\pi G\theta(R_2^4-R_1^4)/l.$$

Since in this case

$$I=\int_{R_1}^{R_2}r^2\,.\,2\pi r\,dr=\tfrac{1}{2}\pi(R_2^4-R_1^4),$$

the formula (6) still holds.

Ex. 1. A steel wire, 1·5 m. long and $\frac{1}{2}$ cm. in diameter, twists 10° when a twisting moment of 60 kg-cm. is applied to its free end. What twisting moment will be required to twist a steel wire 1 cm. in diameter and 60 cm. long through 5°.

Applying the formula $T=\frac{1}{2}\pi G\theta R^4/l$ to the first wire, we get

$$60=\tfrac{1}{2}\pi G\left(\frac{\pi}{18}\right)\left(\frac{1}{4}\right)^4\Big/150. \tag{1}$$

Applying it to the second wire, we get

$$T=\tfrac{1}{2}\pi G\left(\frac{\pi}{36}\right)\left(\frac{1}{2}\right)^4\Big/60. \tag{2}$$

Dividing (2) by (1), we have

$$\frac{T}{60}=\left(\frac{18}{36}\right)\left(\frac{4}{2}\right)^4\frac{150}{60}=20,$$

or
$$T=1200\text{ kg-cm.}$$

Ex. 2. Find the diameter of the shaft, required to transmit 40 horse-power at 300 r. p. m. if the maximum stress is not to exceed 9000 lb./in.²

The angular velocity $=(300/60)2\pi=10\pi$ radians per sec.

As the horse-power transmitted is 40, the work done per second

$$=40\times550\text{ lb.-ft.}$$

This is equal to the product of the twisting moment T and the angle described in one second. Therefore

$$T=\frac{40\times550}{10\pi}=\frac{40\times550\times7}{10\times22}$$
$$=700\text{ lb.-ft.}=8400\text{ lb.-in.}$$

The maximum shear stress f_s occurs at the surface. By formula (5)

$$T=If_s/R=\tfrac{1}{2}\pi R^3f_s.$$

Therefore $8400 = \frac{1}{2}\pi R^3.\ 9000,$

or $R = (84 \times 2/90\pi)^{1/3} = 0{\cdot}84$ inches,

and the required diameter $= 1{\cdot}68$ inches.

EXAMPLES 14

1. The diameter of the piston head of a steam engine is 35 cm. and of piston rod 6 cm. If the engine is working under a steam pressure of 7 kg./sq. cm. what is the maximum compressive stress in the rod ?

2. How much will a 30 metre steel tape 1 cm. wide and $\frac{1}{2}$ mm. thick stretch under a pull of 30 kg. ?

3. A hollow cast iron column 10 feet long has an outside diameter of 10 inches and the metal is 1 inch thick. Find the amount by which the column will be shortened when axially loaded with 50 tons. $E = 8{,}000$ tons per square inch.

[*Roorkee*, 1962]

4. A mild steel rod, 6 m. long, is hung vertically from the ceiling. It has 1·5 cm. diameter for the first 1·5 m. of its length from the top, 2 cm. diameter for the middle 2 m. length, and 2·5 cm. diameter for the lowest 2·5 m. length. Find the largest weight which can be suspended from the lower end of the rod so that the stress in any of its transverse sections does not exceed 1·25 metric tonnes per square cm., determine the magnitude of the weight and the total elongation of the rod. Neglect the weight of the rod. Take E (modulus of elasticity) $= 2200$ tonnes per sq. cm.

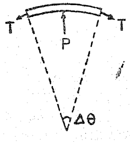
5. A 12 ft. long concrete column having a square cross-section of 1 ft. side is reinforced at the corners by 1 inch steel rods. If E for steel is 12000 tons per square inch and that for concrete 800 tons per square inch, find the load taken up by steel out of a total load of 20 tons on the column. What is the contraction in the column due to the load ? [*Allahabad*, 1965]

6. A steel tube 3 cm. internal diameter, 2·5 mm. thick and 4 m. long is covered and lined throughout with copper tubes 2 mm. thick. The three tubes are firmly fixed at their ends. The compound tube is subjected to tension and stress produced in steel tube is 600 kg./sq. cm. Determine the stress in copper tubes and the total load carried by the compound tube. $E(\text{steel}) = 2{\cdot}1 \times 10^6$ kg./sq. cm., $E(\text{copper}) = 1{\cdot}1 \times 10^6$ kg./sq. cm.

7. In a shaft coupling the radius of the bolt circle is 6″, and there are 6 bolts, each of 1″ diameter. If the permissible shear stress is 4000 lb. per sq. in. what horse-power can be transmitted at 100 r.p.m. ? [*Banaras*, 1955]

8. The shell of a cylindrical boiler is 6 ft. in diameter and the plates are $\frac{3}{8}$ inch thick. It is subjected to an internal fluid

pressure of 160 lb. per sq. in. If the plates on test show an ultimate tensile strength of 30 tons per sq. in., what is the factor of safety allowed ? [*Banaras*, 1959]

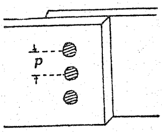


[Hint. If T is the tensile stress in the plates and P ($=160$) the pressure on them, then considering an element of surface of length 1 inch and width $a\Delta\theta$(a=radius of shell=36″), we have

$$P \,.\, a\Delta\theta = 2T(\tfrac{3}{8}) \sin(\tfrac{1}{2}\Delta\theta) = \tfrac{3}{8}T.\Delta\theta,$$

which gives T.]

9. Find the pitch of a single-rivetted lap joint for plates 1·2 cm. thick and rivets 2·2 cm. in diameter. The safe stresses in tension and shear are 700 and 560 kg. per sq. cm. respectively. What is the efficiency of the joint ?



[Hint. The pitch is found from the condition that the shearing stress in the rivets and the tensile stress in the plate between two rivets reach the maximum safe value at the same time. Thus, if p is the pitch and d the diameter of the rivet, both in cm., we have

$$(\tfrac{1}{4}\pi d^2)(560) = \tfrac{1}{2}(p-d)(700).$$

This gives p. The efficiency is the ratio of the total load the joint can carry to the load a plate of the same width (without a joint) could carry. Here, efficiency

$$= \tfrac{1}{2}(p-d)(700)/\{\tfrac{1}{2}p(700)\}$$
$$= (p-d)/p = \text{etc.}]$$

10. Calculate the diameter of a line shafting required to transmit a torque of 5000 kg-cm., if the shearing stress is not allowed to exceed 600 kg. per sq. cm.

11. Find the theoretical diameter of the shaft necessary to transmit a twisting moment of 30 inch-tons, if the maximum shear stress is not to exceed 9000 lb. per sq. in. [*Panjab*, 1958]

12. Two lengths of shaft, each of 5 cm. diameter, are connected by a flanged coupling whose 4 bolts have their centres on a circle concentric with the shaft centre and 20 cm. diameter. Allowing a shear stress of 600 kg./cm.^{2} in the shaft, what is the twisting moment that can be transmitted ? What is the size of bolts if the safe shear stress in them is 350 kg./cm.^{2} ?

13. A solid circular shaft has a diameter of 5 inches and runs at 300 r.p.m. Determine the horse-power transmitted by

the shaft if the maximum allowable stress is limited to 9000 lb./in.^{2}

Calculate the twist, in degrees per foot length of shaft, when it is transmitting the power determined above. Take the modulus of rigidity of the material of the shaft as 12×10^6 lb./in^2.

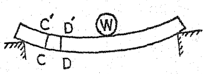
[*Banaras*, 1958]

14. A hollow steel shaft, external diameter d, internal diameter $\frac{3}{5}d$ is subjected to pure twisting, and transmits 8000 horse-power at a speed of 120 r.p.m. Taking the maximum shear stress at 9000 lb. per sq. inch, find d. [*Allahabad*, 1964]

15. A single propeller shaft of solid circular section, 20 in. diameter running in a vessel at 180 r.p.m., is to be replaced by two shafts of hollow circular section, these being equal in diameters and running at 1,000 r.p.m., but developing 60 per cent more horse-power. The internal diameter of each new shaft is to be one-half of the external diameter. Find the external and internal diameters of the new shafts if the maximum working stress in these shafts may be taken as 25 per cent greater than the same in the old shaft. [*Banaras*, 1959]

9·4. Beams and bending. A bar subjected to external forces which have components normal to the bar is called a *beam*. The normal components of the external forces induce a bending of the beam. Quite often in practice the beams are horizontal, and the external forces are the vertical weights of various loads and the reactions at the supports. In the present chapter we shall consider only straight beams of uniform cross-section. We shall further suppose that the beams are horizontal and the loads acting on them are vertical.

A simple case of bending is that of a beam resting on two supports at its ends and loaded by a weight W at its middle point. After bending the beam will be convex downwards, and any two lines CC' and DD' originally vertical and parallel, will no longer be parallel. It will be found that due to bending $C'D'$ will get contracted and CD will get extended. Similarly in more

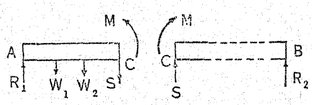


also, some elements of the beam get contracted and are in compression while others get stretched and are in tension. Some elements in the middle will be neither compressed nor stretched. The surface containing all such elements which are unaltered in length due to bending is called the *neutral surface* of the beam.

When investigating the forces in a beam, the first step generally is to determine the reactions at the supports. This can be done by the methods of chapter III. The two types of beams commonly met in practice are the simply supported beam and the cantilever beam. A beam is said to be *simply supported* when it rests on two supports in such a manner that the reactions at the supports are normal to the beam. A *cantilever* beam is built in or fixed (in a wall or to another part of the structure) at one end and is free at the other end. The reaction at the fixed end consists of a normal reaction and a couple (see Ex. 2 p. 39).

9·5. Shearing Force and Bending Moment. Consider a beam AB supported at the ends and carrying a number of loads $W_1, W_2, \ldots$, at distances $a_1, a_2, \ldots$ from the end A. Let R_1 and R_2 be the reactions of the supports. Then R_1 and R_2 can be found from the condition of equilibrium of the beam.

Take a section of the beam at C distant x from A, and consider the equilibrium of the part AC. This will be under the action of the external forces R_1, W_1, W_2 and the internal forces at C exerted by the part CB on the part AC.



Since the algebraic sum of the vertical upward external forces on AC is

$$R_1 - W_1 - W_2,$$

Again, since the algebraic sum of the moments of external forces about C in clockwise direction is

$$R_1x-W_1(x-a_1)-W_2(x-a_2),$$

there must act at C a couple of the above magnitude in the anticlockwise direction. The action of CB on AC thus consists of a vertical force together with a couple. The latter arises due to longitudinal compression and tension in the upper and lower layers of the beam respectively.

The vertical force acting at C (which is the resultant of the tangential forces on the cross-section) is called the *shearing force* at C; and the moment of the normal forces on the cross-section is called the *bending moment* at C. We shall denote the former by S and the latter by M.

The conditions of equilibrium of the part AC give

$$S=R_1-W_1-W_2 \quad . \quad . \quad . \quad (1)$$

= the algebraic sum of the transverse components of the external forces acting in the upward direction on the part AC of the beam;

$$M=R_1x-W_1(x-a_1)-W_2(x-a_2) \quad . \quad . \quad . \quad (2)$$

= the algebraic sum of the moments in the clockwise direction of the external forces acting on the part AC of the beam.

Since the action and reaction at C are equal and opposite, therefore the shearing force and the bending moment at C may also be calculated from the part CB of the beam, only the signs should be reversed.

We shall take the shearing force in upward direction and the bending moment in the clockwise direction as positive when they are acting at the left end of a member. When they are acting at the right end, the opposite directions will be taken as positive. The figure above shows the positive directions of S and M acting on AC and CB.

We see from (1) and (2) that

$$S=dM/dx.$$

An independent derivation of this result will be given in the next

The manner in which the shearing force and bending moment vary from point to point along a beam is usually shown by drawing shearing force and bending moment diagrams beneath a sketch of the beam, as in the following examples.

Ex. 1. Draw the shearing force and the bending moment diagrams for a simple beam of length l feet when

(i) carrying a load W pounds at its middle point; [*Roorkee*, 1962]

(ii) carrying a uniformly distributed load of w pounds per foot. [*Allahabad*, 1965]

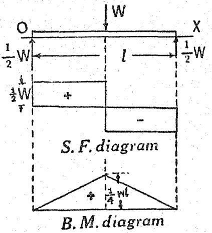
In each case find the maximum bending moment.

A simple beam is a beam simply supported at its ends. Take the origin at the left end.

(i) The reactions of the supports are

$$R_1 = R_2 = \tfrac{1}{2}W.$$

For $0 < x < \frac{1}{2}l$, the only external force on the left portion is the reaction $\frac{1}{2}W$. Therefore



(a)

$$S = \tfrac{1}{2}W,$$

$$M = \tfrac{1}{2}Wx.$$

For $\frac{1}{2}l < x < l$, the external forces acting to the left of x are the upward reaction $\frac{1}{2}W$ at O, and the downward load W at $x = \frac{1}{2}l$. Therefore

$$S = \tfrac{1}{2}W - W = -\tfrac{1}{2}W,$$

$$M = \tfrac{1}{2}Wx - W(x - \tfrac{1}{2}l) = \tfrac{1}{2}W(l - x).$$

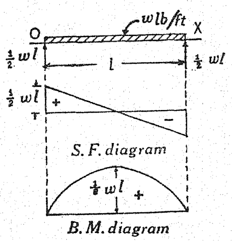
The above values of S and M are plotted in the shearing force and bending moment diagrams in figure (*a*).

$$M_{max} = \tfrac{1}{4}Wl \text{ lb.-ft.}$$

(ii) The reactions of the supports are

$$R_1 = R_2 = \tfrac{1}{2}wl.$$

The external forces acting on the left portion of length x are the upward reaction $\frac{1}{2}Wl$ at O, and the weight wx acting downwards



at a distance $\frac{1}{2}x$ from O. These give, for a point distant x from O,

$$S=\tfrac{1}{2}wl-wx,$$

$$M=\tfrac{1}{2}wlx-\tfrac{1}{2}wx^2.$$

These values are plotted in figure (b).

$$M_{max}=\frac{wl^2}{8} \text{ lb.-ft.}$$

Ex. 2. Draw the shearing force and the bending moment diagrams for a cantilever beam, of length l feet, when

(i) carrying a load of W pounds at the free end;

(ii) carrying a distributed load of w lb./ft. [*Roorkee*, 1964]

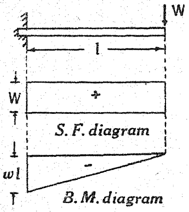
In each case find the maximum bending moment.

For a cantilever beam, besides the vertical reaction, there would be a couple also at the fixed end. Therefore in this case take origin at the free end, and consider the right portion of the beam.

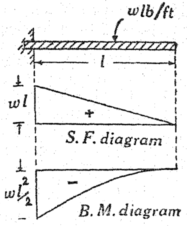
(i) For $0<x<l$, $S=W$, $M=-Wx$.

From these values of S and M, the S. F. and B. M. diagrams are drawn as shown in figure (a).

$$M_{max}=-Wl \text{ lb.-ft.}$$



(a)



(b)

(ii) for $0<x<l$, $S=wx$, $M=-\frac{1}{2}wx^2$.

These values of S and M are plotted in figure (b).

$$M_{max}=-\tfrac{1}{2}wl^2.$$

Ex. 3. A beam 20 feet long, resting on two supports 4 feet from either end, carries a uniformly distributed load of 1 ton per foot. Draw shearing force and bending moment diagrams.

The reactions of the supports are $R_1 = R_2 = 10$ tons.

For $0 < x < 4'$,

$$S = -x,$$

$$M = -\tfrac{1}{2}x^2.$$

For $4' < x < 16'$,

$$S = 10 - x,$$

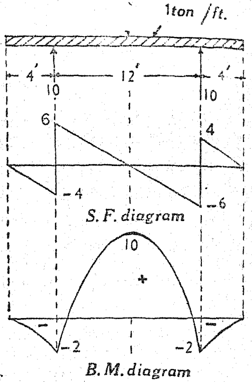
$$M = 10(x-4) - \tfrac{1}{2}x^2.$$

For $16' < x < 20'$,

$$S = (20 - x),$$

$$M = 10(x-4) + 10(x-16) - \tfrac{1}{2}x^2.$$

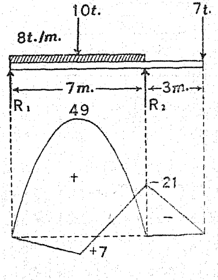
$$= 20x - 200 - \tfrac{1}{2}x^2.$$



The shearing force and bending moment diagrams are now drawn from these values for the various intervals.

Ex. 4. A beam, length 10 metres, is supported at the left end and at a point 7 m. from it. It carries a distributed load of 8 tonnes per metre from the left end for 7 metres and two loads of 10 tonnes and 7 tonnes at 3·5 m. and 10 m. from the left end. Draw the bending moment diagram.

The bending moment M at any point is the sum of the bending moments due to the distributed load and the concentrated loads. To simplify the problem we draw the B.M. diagram for the distributed load and superpose over it the B.M. diagram for the concentrated loads.



For the distributed load:

Reaction $\quad R_1 = R_2 = 28$ tonnes.

For $0 < x < 7$,

$$M = 28\,x - 4x^2.$$

For the concentrated loads:

$R_1 = 2$ tonnes, $R_2 = 15$ tonnes.

For $\quad 0 < x < 3{\cdot}5,\ M = 2x.$

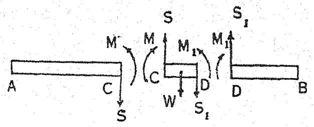
For $\quad 3{\cdot}5 < x < 7,\ M = 2x - 10(x - 3{\cdot}5)$

For $7 < x < 10$,

$$M = 2x - 10(x - 3\cdot5) + 15(x - 7) = 7x - 70.$$

From above equations the B.M. diagram for the distributed load is drawn and B. M. diagram for the concentrated loads superposed in such a way that the height from one line to the other gives the bending moment at that point.

9·51. Relation between S and M. Consider the equilibrium of a small element CD of a beam AB. The forces acting on this element are shown in the adjoining figure. Resolving vertically and taking moments about C, we get



$$S_1 - S + W = 0, \tag{1}$$

and
$$M_1 - M = S_1 \,.\, CD + W \,.\, \tfrac{1}{2}CD. \tag{2}$$

We now consider the following three cases:

(i) If $W = 0$, i.e., if the element CD does not carry any load, then by (1) $S_1 = S$; and by (2), on dividing by CD and taking limits as CD $(= \delta x)$ tends to zero,

$$\frac{dM}{dx} = S. \quad . \quad . \quad . \tag{3}$$

(ii) If the element CD carries a point load W, then $S_1 = S - W$ and the shearing force increases discontinuously. From (2) we get, as $CD \to 0$, $M_1 \to M$. Thus the bending moment is continuous, but its gradient is not continuous.

(iii) If the element CD carries a uniform load w per unit length, then $W = w \,.\, CD$, and (1) gives

$$S_1 - S = -w \,.\, CD.$$

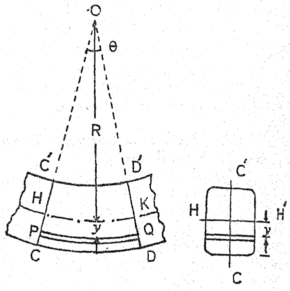
Dividing by CD and taking the limit as CD $(= \delta x)$ tends to zero, we get

$$\frac{dS}{dx} = -w. \quad . \quad . \quad . \tag{4}$$

Equation (2) leads in this case also to the relation $dM/dx = S$.

9·6. Stresses induced by bending. Let a beam AB undergo bending due to transverse loads on it. It will be assumed

the element $CC'D'D$ of the beam. Let HK be the neutral surface. Due to bending the sections CC' and DD', though remaining plane, become slightly inclined to one another. Suppose they meet on being produced at O where $OH=R$. Then HK which is normal to both these sections, becomes a part of a circle of radius R. Similarly, a layer PQ at depth y below HK, becomes a part of a circle of radius $R+y$.



The strain in PQ is given by

$$\frac{PQ-HK}{HK}=\frac{(R+y)\theta-R\theta}{R\theta}=\frac{y}{R},$$

where θ denotes the angle COD. Hence the stress in PQ is

$$Ey/R. \quad . \quad . \quad . \quad (1)$$

If the cross-sectional area of the layer PQ is δa, then the tensile force acting over this area in direction PQ is $(Ey/R)\delta a$. The moment of this force about the neutral axis HH' is

$$(Ey^2/R)\delta a.$$

Summing up for all the layers in the element $CC'D'D$, we see that the total moment of the tensile and compressive forces about HH' is

$$(E/R)\Sigma y^2 \delta a.$$

This is the moment set up by the internal forces at the section CC' and gives the bending moment M. Hence

$$M=(E/R)\Sigma y^2 \delta a.$$

The expression $\Sigma y^2 \delta a$ is known as the 'moment of inertia' of the section about the neutral axis. This can be easily found by integration (see Part II). It is generally denoted by I. Thus

$$M=EI/R. \quad . \quad . \quad . \quad (2)$$

It is seen from (2) that the radius R remains constant only when the bending moment M is constant for all the elements of the beam. In this case $S=dM/dx=0$, and the shearing forces are absent. Such a case is called *pure bending*, and can be induced by couples applied to the two ends of the beam. In the general case of loading M is not constant and R

9·61. Location of neutral axis. We have seen in §9·5, that the internal action at a section consists of a transverse force together with a couple. Hence the algebraic sum of the longitudinal forces across a section must be zero. This gives, by (1) of preceding section,

$$\Sigma(Ey/R)\delta a=0. \qquad (3)$$

Since the coordinate of the centre of gravity $\bar{y}=\Sigma y\delta a/\Sigma\delta a$, relation (3) means that $\bar{y}=0$. Therefore the neutral axis is a horizontal line passing through the centre of gravity of the section. For a symmetrical section it will pass through the middle of the section.

9·62. Strength of beams. We have seen in § 9·6 [equation (1)] that the stress in a layer of the beam at depth y below the neutral surface is Ey/R. The maximum stress will therefore occur at the lower and upper surfaces of the beam. If f be the maximum stress and D the depth of the beam, then for a beam with symmetrical section,

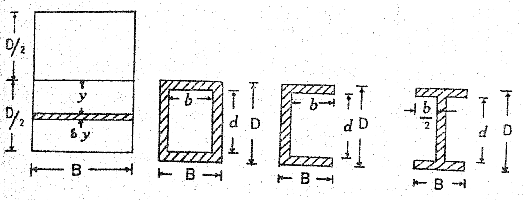
$$f=\frac{ED}{2R}=\frac{MD}{2I}, \text{ by (2).} \qquad (4)$$

For a rectangular section of breadth B and depth D (fig. a),

$$I=\Sigma y^2\delta a=\int_{-D/2}^{D/2} y^2 \,.\, B\,dy$$
$$=BD^3/12.$$

The sections shown in figure (b) can all be considered as composed of a rectangle of breadth B and depth D. from which a smaller rectangle of breadth b and depth d has been removed. Therefore, in these cases

$$I=\Sigma y^2\delta a$$
$$=(BD^3-bd^3)/12.$$



(a) (b)

It can be shown similarly that for a circular section of diameter D, $I=\pi D^4/64$. If the section be annular, with inner and outer radii d and D, $I=\pi(D^4-d^4)/64$.

The strength of a beam is measured by the greatest bending moment to which it may safely be subjected. If in a given case the maximum stress which the material can bear is known then by calculating the maximum bending moment, the dimensions of the beam can be obtained for safe loading.

Ex. 1. A wooden beam, 12 feet long, is simply supported at its ends and carries a uniformly distributed load of 350 lb./ft. If the greatest stress is limited to 900 lb./in.^{2}, find the dimensions of the beam, if the width is half the depth.

The maximum bending moment occurs at the middle point of the beam (Ex. 1, § 9·5), and is

$$\tfrac{1}{8}wl^2=\tfrac{1}{8}(350\times 12^2)=6300 \text{ lb.-ft.}$$
$$=75600 \text{ lb.-in.}$$

From formula (4) we have

$$f=\frac{MD}{2I}=\frac{12MD}{2BD^3}=\frac{6M}{BD^2},$$

or $$BD^2=6M/f=6\times 75600/900=504 \text{ in.}^3$$

Since $B=\frac{1}{2}D$, therefore

$$D^3=1008,$$

or $$D=10 \text{ inches, approximately.}$$
$$B=\tfrac{1}{2}D=5 \text{ inches.}$$

Ex. 2. The roof of a room weighs 400 kg./m.^{2} and is supported on beams, spaced $1\frac{1}{4}$ m. apart, centre to centre. The beams are placed on walls 4 m. apart. If the ultimate strength of wood is 550 kg./cm^2. find the dimensions of the beams, taking the factor of safety as 10, and the width equal to 0·6 times the depth.

The weight per unit length of the beam $w=400\times\frac{5}{4}=500$ kg./m. The maximum bending moment due to this load is at the middle point, and is

$$\tfrac{1}{8}wl^2=\tfrac{1}{8}(500\times 4^2)=1000 \text{ kg.-m.}=10^5 \text{ kg.-cm.}$$

Since the factor of safety is 10, the maximum permissible stress $f=550/10=55$ kg./cm. As in Ex. 1, we have,

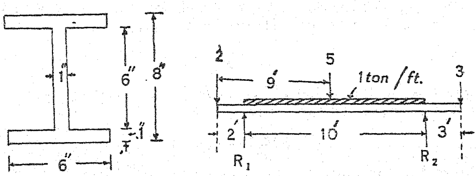
$$BD^2=6M/f=6\times 10^5/55=10909 \text{ cm}^3.$$

Since $B=0{\cdot}6D$, therefore $D^3=18182$, or $D=26{\cdot}3$ cm.

Hence the dimensions may be taken as 26·5 cm. $\times$ 16 cm.

Ex. 3. A steel beam, 15 feet long, cross-section as shown, is supported at 2 feet from the left end and 3 feet from the right end. It carries a distributed load of 1 ton per foot between the supports and three loads of 2 tons, 3 tons and 5 tons at the left end,

right end and at 9 feet from left end respectively. Find the maximum stress in the fibres of the beam.



Let R_1 and R_2 be the reactions at the supports, then taking moments about the left support,

$$R_2=\tfrac{1}{10}(5\times 7+3\times 13+10\times 5-2\times 2)=12 \text{ tons.}$$

$$R_1=2+5+3+10-R_2=8 \text{ tons.}$$

The bending moment M is maximum at the point where $S=0$ or at the supports. In the interval $2<x<9$

$$S=8-2-(x-2), \qquad \ldots (1)$$

$$M=8(x-2)-2x-\tfrac{1}{2}(x-2)^2. \qquad \ldots (2)$$

From (1) $\quad S=0$ at $x=8$.

$\therefore$ maximum $\quad M=8(8-2)-2\times 8-\tfrac{1}{2}\times 36=14$ ton-ft.

In the interval $9<x<12$,

$$S=1-2-5-(x-2)=3-x.$$

So M does not have any maximum in this interval. At the supports $M<14$ ton-ft. Hence the maximum bending moment is 14 ton-ft.

The moment of inertia of the section $I=(BD^3-bd^3)/12$

$$=(6\times 8^3-5\times 6^3)/12=166 \text{ in.}^4$$

From formula $f=MD/2I$, we get

$$f=\frac{14\times 2240\times 12\times 8}{2\times 166}$$

$$=9067 \text{ lb./in.}^2$$

9·7. Equation of the elastic curve. Let a beam AB be loaded transversely, and let a point on the neutral surface distant x from the end A be displaced transversely through y due to bending. Then the relation between x and y will give the equation of the curve which the neutral surface assumes on bending.

We know from calculus that the radius of curvature R of a curve at a point (x, y) is given by

$$R=\frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2}.$$

In actual practice the slope dy/dx of the elastic curve is very small and its square is negligible. Hence the curvature can be taken as

$$\frac{1}{R}=\pm\frac{d^2y}{dx^2}.$$

Putting this in the equation $M=EI/R$ for the bending moment, and taking the positive sign, we get

$$EI\frac{d^2y}{dx^2}=M, \qquad . \quad . \quad . \quad (1)$$

which is the differential equation of the elastic curve.

Integrating (1) twice, we get

$$EIy=\int\{\int Mdx\}dx+C_1x+C_2.$$

If the beam is resting on two supports at its ends, the constants C_1, C_2 can be determined from the conditions that $y=0$ at $x=0$ and $x=l$. If it is a cantilever the conditions are that $y=0$ and $dy/dx=0$ at $x=0$. Similar conditions will determine the constants of integration for any other beam.

Taking $1/R=+d^2y/dx^2$ implies that R is in the direction of y increasing. Hence in (1), with the convention of sign already adopted for the bending moment M, y must be measured upwards.

We also notice from equation (1) that $d^2y/dx^2=0$ at the points where $M=0$. Such points are points of inflexion for the elastic curve. The curvature changes from concavity to convexity, or vice versa at these points.

Ex. Find the equation of the elastic curve and the maximum deflection for a simple beam, l feet long, which carries (i) a load W at the mid-point of the beam (ii) a uniformly distributed load w per foot.

The differential equation of the elastic curve is:

$$\frac{d^2y}{dx^2}=\frac{M}{EI}.$$

Due to symmetry, the end conditions may be taken as

$$x=0 \text{ at } y=0 \text{ and } \frac{dy}{dx}=0 \text{ at } x=\tfrac{1}{2}l.$$

(i) As in Ex. 1, § 9·5, for $0\leqslant x\leqslant\frac{1}{2}l$,

$$M=\tfrac{1}{2}Wx.$$

Therefore
$$\frac{d^2y}{dx^2}=\frac{W}{2EI}x.$$

Integrating once, we get

$$\frac{dy}{dx}=\frac{W}{2EI}\left(\frac{x^2}{2}+A\right).$$

Since $dy/dx=0$ when $x=\frac{1}{2}l$, therefore $A=-\frac{1}{8}l^2$. Hence

$$\frac{dy}{dx}=\frac{W}{4EI}\left(x^2-\frac{l^2}{4}\right).$$

Integrating again, we have

$$y=\frac{W}{4EI}\left(\frac{x^3}{3}-\frac{l^2}{4}x+B\right).$$

Since $x=0$ when $y=0$, therefore $B=0$.

Hence the equation of the elastic curve $(0\leqslant x\leqslant\frac{1}{2}l)$ is

$$y=\frac{W}{4EI}\left(\frac{x^3}{3}-\frac{l^2}{4}x\right).$$

For $\frac{1}{2}l<x\leqslant l$, the curve can be obtained from symmetry. Maximum deflection occurs at $x=\frac{1}{2}l$, therefore

$$y_{max}=\frac{W}{4EI}\left[\frac{x^3}{3}-\frac{l^2}{4}x\right]_{x=l/2}=-\frac{Wl^3}{48EI}.$$

(ii) As in Ex. 1, § 9.5,

$$M=\tfrac{1}{2}wlx-\tfrac{1}{2}wx^2.$$

Therefore
$$\frac{d^2y}{dx^2}=\frac{w}{2EI}(lx-x^2).$$

Integrating once, we get

$$\frac{dy}{dx}=\frac{w}{2EI}\left(\frac{lx^2}{2}-\frac{x^3}{3}+A\right).$$

When $x=\frac{1}{2}l$, $dy/dx=0$, therefore $A=-l^3/12$. Hence

$$\frac{dy}{dx}=\frac{w}{2EI}\left(\frac{lx^2}{2}-\frac{x^3}{3}-\frac{l^3}{12}\right).$$

Integrating again, we have

$$y=\frac{w}{2EI}\left(\frac{lx^3}{6}-\frac{x^4}{12}-\frac{l^3}{12}x+B\right).$$

When $x=0, y=0$, therefore $B=0$.

Hence the equation of the elastic curve is

$$y=\frac{w}{12EI}\left(lx^3-\frac{x^4}{2}-\frac{l^3x}{2}\right).$$

Maximum deflection occurs at $x=\frac{1}{2}l$, therefore

$$y_{max}=\frac{w}{12EI}\left[\frac{l^4}{8}-\frac{l^4}{32}-\frac{l^4}{4}\right]=-\frac{5wl^4}{384EI}.$$

Examples 15

1. A timber beam, 6 metres long, is supported freely at both ends and carries a concentrated load of 2 tonnes at the centre of the span. If the beam is rectangular in section, the depth being twice the breadth, and the stress in the timber is limited to 50 kg. per sq. cm., determine the cross-section of the beam.

2. A rectangular beam 15 cm. $\times$ 20 cm. is freely supported at two points 3 m. apart and carries a uniformly distributed load of 2000 kg. per metre between the supports. Draw to scale the shearing force and bending moment diagrams and determine the maximum stress in the beam.

3. A uniform rod AB is supported at each end. If w be its weight per unit length, prove that the bending moment at any point P is $\frac{1}{2} w . AP . PB$. [*Roorkee*, 1966]

4. A rectangular beam 2″ wide, 3″ deep and 15 ft. long is simply supported at its ends and carries loads of 150 lb. and 100 lb. at distances of 2 ft. and 12 ft. respectively from the left end. Determine the maximum stress and state where it occurs. [*Banaras*, 1956]

5. A cantilever 8 cm. $\times$ 12 cm. and 1·5 m. long has to carry a uniformly distributed load over its entire length. Find the value of the load per foot run if the maximum stress is not to exceed 90 kg. per sq. cm.

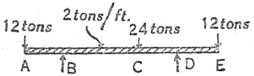
6. A timber cantilever 3″ wide, 5″ deep and 10′ long carries a load of 75 lb. at a distance of 6 ft. from the fixed end and a load of 60 lb. at the free end. Draw the shearing force and bending moment diagrams and determine the maximum stress produced. Neglect the weight of the cantilever. [*Banaras*, 1955]

7. A simple beam 3 m. long has a rectangular cross-section 20 cm. deep. It carries loads of 4 tonnes and 3 tonnes at 1 m. and 2 m. from the left end. Find the width for safe beam if fibre stress is limited to 80 kg. per sq. cm.

8. A beam 10 ft. long is simply supported at both ends. It carries a load of 2 tons at distance 2 ft. from the left end and a load of 5 tons at distance 3 ft. from the right end. Draw to scale, giving the main dimensions, the bending moment and shearing force diagrams for the beam.

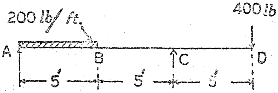
If an additional load of 1 ton per foot run is uniformly distributed over the beam, find the position and magnitude of the maximum bending moment. What is the value of the shearing force under this point of maximum bending moment? [*Banaras*, 1959]

9. Draw the diagram of B.M. and S.F. for a beam loaded as shown. Uniform load $=2$ tons per foot run.



$AB=\frac{1}{2}BC=CD=DE=4$ ft. [*Panjab*, 1957]

10. Draw the bending moment and shear force diagrams for the beam loaded as shown in the figure below. [*Panjab*, 1958]



11. A light rod of length $2l$ rests symmetrically on two rigid supports at a distance $2a$ apart. If a load W is suspended from the centre, show that this point will sink through a distance $Wa^3/6EI$ and the ends will rise through a distance $Wa^2(l-a)/4EI$.

[*Roorkee*, 1967]

12. Find the equation of the elastic curve and the maximum deflection for a cantilever beam of length l fixed at the end $x=0$, which carries

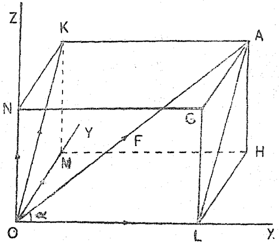
(i) a load W at the end $x=l$;

(ii) a uniformly distributed load w per unit length.

CHAPTER X

FORCES IN SPACE

10·1. Components along three rectangular axes. Let OA represent a force F in space and let α, β and γ be the direction angles of the vector OA with regard to a system of three mutually perpendicular axes OX, OY and OZ.



With OA as diagonal, construct a parallelepiped ($OLHM$, $NGAK$) by drawing planes parallel to the coordinate planes, as shown in the figure. Join AL and OK.

Referring to the parallelogram $OLAK$, we find that

$$\overline{OA} = \overline{OL} + \overline{OK}.$$

Similarly from the parallelogram $OMKN$, we get

$$\overline{OK} = \overline{OM} + \overline{ON}.$$

Thus $$\overline{OA} = \overline{OL} + \overline{OM} + \overline{ON}.$$

Also from the triangle OAL,

$$OL = OA \cos \alpha = F \cos \alpha.$$

Similarly, $OM = F \cos \beta$, and $ON = F \cos \gamma$.

Hence a given force F in space can be resolved into three components $F \cos \alpha$, $F \cos \beta$ and $F \cos \gamma$ along the three axes. If the direction cosines $\cos \alpha$, $\cos \beta$, $\cos \gamma$ are respectively denoted by l, m, n the components may also be written as Fl, Fm and Fn.

Conversely, if three forces X, Y, Z act along the axes of x, y and z respectively, their resultant F is given by

$$F=\sqrt{(X^2+Y^2+Z^2)},$$

and angles α, β, γ which the resultant makes with the axes, are given by

$$\cos\alpha=\frac{X}{F},\ \cos\beta=\frac{Y}{F},\ \cos\gamma=\frac{Z}{F}.$$

10·2. Resultant of concurrent forces in space. Let F_1, F_2, F_3, ... be a system of concurrent forces acting at O.

Take a set of three rectangular axes through O, and let the direction cosines of the forces F_1, F_2, F_3, ... be l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3; ... respectively.

Resolve each force of the system into x-, y-, and z- components. The components along each axis constitute a system of collinear forces and hence can be added algebraically. Thus we have the three forces:

$$X=F_1l_1+F_2l_2+F_3l_3+\ldots,$$
$$Y=F_1m_1+F_2m_2+F_3m_3+\ldots,$$
$$Z=F_1n_1+F_2n_2+F_3n_3+\ldots,$$

acting along the axes of x, y and z respectively.

These three forces can be combined into a single resultant R whose line of action makes the angles α, β, γ with the axes where

$$R=\sqrt{(X^2+Y^2+Z^2)},$$

and $$\cos\alpha=\frac{X}{R},\ \cos\beta=\frac{Y}{R},\ \cos\gamma=\frac{Z}{R}.$$

Ex. Find the resultant of the system of forces acting on a body as givn below :—

(i) 80 kg. from origin towards the point (0, 5, 5),
(ii) 130 kg. from origin towards the point (—3, 4, 12),
(iii) 85 kg. from origin towards the point (8, 0, —15),
(iv) 100 kg. from (8, 6, 0) towards the origin.

The distance from the origin to the point (0, 5, 5) is $\sqrt{(5^2+5^2)} = 5\sqrt{2}$. Hence the direction cosines of the line from origin to (0, 5, 5) is

$$0, \frac{5}{5\sqrt{2}}, \frac{5}{5\sqrt{2}}, \text{ i.e. } 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}.$$

The components of the 80 lb. force are therefore 0, $80/\sqrt{2}$, $80/\sqrt{2}$. Similarly, the components of the other forces can be obtained. It is convenient to tabulate the solution as below :

Forces	Direction cosines	X	Y	Z
80 kg.	$0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$	0	56·6	56·6
130 kg.	$-\frac{3}{13}, \frac{4}{13}, \frac{12}{13}$	−30	40	120
85 kg.	$\frac{8}{17}, 0, -\frac{15}{17}$	40	0	−75
100 kg.	$-\frac{4}{5}, -\frac{3}{5}, 0$	−80	−60	0
90 kg.	1, 0, 0	90	0	0
		20	36·6	101·6

Therefore the magnitude of the resultant R

$$= \sqrt{\{20^2+(36{\cdot}6)^2+(101{\cdot}6)^2\}} = 109{\cdot}8 \text{ kg.},$$

and its direction is given by

$\cos\alpha = 20/109{\cdot}8 = {\cdot}1822$, or $\alpha = 79° \ 30'$.

$\cos\beta = 36{\cdot}6/109{\cdot}8 = {\cdot}3334$, or $\beta = 70° \ 30'$.

$\cos\gamma = 101{\cdot}6/109{\cdot}8 = {\cdot}9253$, or $\gamma = 22° \ 18'$.

10·3. Equilibrium of concurrent forces in space. We have seen that the resultant of a system of concurrent forces in space is a single force R given by

For the equilibrium of the system, R should vanish. But $R=0$ if and only if,

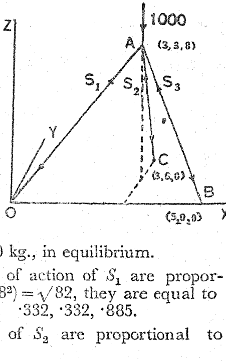
$$X=0,\ Y=0,\ Z=0. \qquad (1)$$

Thus a necessary and sufficient set of conditions for a system of concurrent forces to be in equilibrium is that the sum of the components of all the forces along each axis should be separately zero.

The equations (1) are used to determine unknown elements in a problem of concurrent forces in equilibrium. As there are three independent equations of equilibrium, not more than three unknown elements can be determined.

Sometimes, specially when there is a symmetry, it is convenient to reduce the system to a set of coplanar forces, by combining two forces into a single force, as in Ex. 2 below. The problem can then be solved by an application of the methods for coplanar forces.

Ex. 1. A tripod with legs AO, AB, AC rests on level ground as shown in the figure. The points of support O, B and C on the ground have the coordinates (0, 0, 0), (5, 0, 0) and (3, 6, 0) respectively, and the point A has the coordinates (3, 3, 8). A load of 1000 kg. acts downwards at the hinge A. Determine the magnitudes of the stresses in the legs caused by the load.



Let S_1, S_2 and S_3 be the forces (compressions) in the members AO, AC and AB respectively. Hence the free-body diagram for the point A gives four forces, S_1, S_2, S_3 and 1000 kg., in equilibrium.

The direction cosines of the line of action of S_1 are proportional to $3:3:8$. Since $\sqrt{(3^2+3^2+8^2)}=\sqrt{82}$, they are equal to

$$3/\sqrt{82},\ 3/\sqrt{82},\ 8/\sqrt{82}, \quad \text{i.e.} \quad \cdot 332,\ \cdot 332,\ \cdot 885.$$

Similarly, the direction cosines of S_2 are proportional to $3-3:3-6:8$; and are equal to

$$0,\ -3/\sqrt{73},\ 8/\sqrt{73}, \quad \text{i.e.} \quad -0,\ -\cdot 352,\ \cdot 938.$$

The direction cosines of S_3 are proportional to $3-5:3:8$; and are equal to

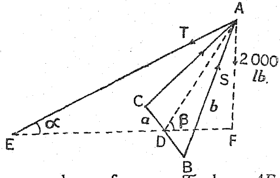
Hence, equating to zero the sum of the components of the forces along x-, y- and z-axis respectively, we get

$$X = \cdot 332\, S_1 - \cdot 228\, S_3 = 0,$$
$$Y = \cdot 332\, S_1 - \cdot 352\, S_2 + \cdot 342\, S_3 = 0.$$
$$Z = \cdot 885\, S_1 + \cdot 938\, S_2 + \cdot 913\, S_3 - 1000 = 0.$$

Solving these, we obtain,

$$S_1 = 225{\cdot}5 \text{ kg.}, \; S_2 = 532{\cdot}5 \text{ kg.}, \; S_3 = 329 \text{ kg.}$$

Ex. 2. A shear legs crane has legs AB and AC, each of length b and hinged at A, B and C as shown in the figure. They are held in position by a stay cable AE in the vertical plane through AD, which is a median of the triangle ABC. A weight of 2000 lb. hangs from A. If $BD = CD = a$, angle $AED = \alpha$ and angle $ADF = \beta$, find the stress in the legs AB and AC and the tension in the stay cable AE.



Let T be the tension in AE and S the compression in the leg AB. Then by symmetry, the compression in the leg AC is also S. The resultant of these two forces in AB and AC is $2S \cos BAD$ acting along DA. Thus the given system is equivalent to the three coplanar forces : T along AE, $2S \cos BAD$ along DA, and 2000 lb. along the vertical AF. As these are in equilibrium, we have by Lami's theorem

$$\frac{T}{\sin(90+\beta)} = \frac{2S \cos BAD}{\sin(90-\alpha)} = \frac{2000}{\sin(\beta-\alpha)}.$$

also $\quad \cos BAD = AD/AB = \sqrt{(b^2-a^2)}/b.$ Therefore

$$T = \frac{2000 \cos\beta}{\sin(\beta-\alpha)} \text{ lb.}$$

and

$$S = \frac{1000\, b \cos\alpha}{\sqrt{(b^2-a^2)} \,.\, \sin(\beta-\alpha)} \text{ lb.}$$

Examples 16

1. Three forces of 20, 30 and 40 kg. meet at a point O and are directed away from O along the diagonals of three faces of a cube meeting at the point O. Find the magnitude of their resultant and the angles the resultant makes with the three edges of the cube.

2. Three forces of magnitudes 30 kg., 50 kg. and 80 kg. act at the point O in the directions OA, OB and OC respectively. The angle AOB is 60°, the angle AOC is 45° and the angle BOC is 30°. Find the magnitude of the resultant.

[Hint. Take the x-axis along OA, and y-axis in the plane AOB.]

3. If P_1, P_2, P_3, ... be a set of non-coplanar forces acting at a point, show that the resultant force R is given by magnitude

$$R^2 = \Sigma P_r^2 + 2\Sigma P_r P_s \cos\theta_{rs}. \qquad [Banaras, 1950]$$

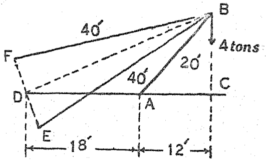
4. A theodolite, weighing 5 pounds, rests at the top of a tripod with legs 6 feet long, which are placed at the vertices of an equilateral triangle on a horizontal floor. The top of the tripod is 5 feet from the floor. Find the force of compression on each leg.

5. A weight of 80 pounds is hung from a horizontal ring 6 feet in diameter by means of three cords, each 5 feet long. On the ring two cords are 90° apart and the point of attachment of the third cord bisects the remaining arc of the ring. Find the tension in each cord. [*Banaras*, 1954]

6. Three wires, each 3 m. long, are attached with their upper ends to a horizontal ceiling. The points of attachment form an equilateral triangle of side 3 m. The lower ends are jointed together and a load of 100 kg. is hung from the joint. Calculate the tension in each wire.

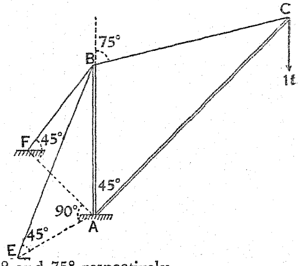
7. A 20-feet spar AB, used as a derrick, is supported by two back stays EB and FB, each 40 feet long as shown in the figure.

If $DA = 18$ feet and $AC = 12$ feet, find the compression in AB and the tension in each back-stay, when a weight of 4 tons hangs from B.



[*Banaras*, 1957]

8. The vertical mast AB of a crane rests in a socket at A on the ground and is supported by two guy wires BE and BF each inclined at 45° to the horizontal, E and F being on the ground. The plane ABC of the crance bisects the angle EAF which is 90°. Find the tensions in the guy wires and the tie BC, and the thrust in the jib AC when a load of 1 tonne is suspended from C. The angles AC and BC make with the vertical are 45° and 75° respectively.



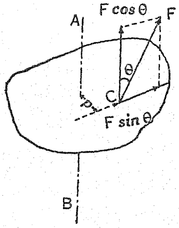
9. Three vertical poles 30 ft., 20 ft. and 25 ft. in height are erected on level ground at points whose coordinates, on the ground plane, are (0, 0), (0, 30 ft.) and (40 ft., 10 ft.) respectively. A load of 700 pounds is suspended from point A, 10 ft. above the ground, by means of cables running from A to the top of each pole. The line of action of the load passes through a point on the ground plane, having coordinates (20 ft. 10 ft.). Find the tension in each cable. [*Banaras*, 1953]

10. A heavy particle is placed on a rough plane, inclined at an angle α to the horizontal, and is connected by a weightless string AP to a fixed point P in the plane. If AB is the line of greatest slope and θ the angle PAB when the particle is on the point of slipping show that

$$\sin \theta = \mu \cot \alpha,$$

where μ is the coefficient of friction between the particle and the plane.

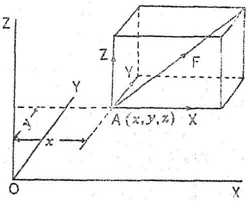
10·4. Moment of a force about an axis. The moment of a force about an axis is the measure of the tendency of the force to produce rotation of the body on which it is acting about that axis.



If AB is the axis and F the force making an angle θ with AB, then F can be resolved into a component $F \cos \theta$ parallel to AB and a component $F \sin \theta$ perpendicular to AB. The component $F \cos \theta$ has no tendency to turn the body about AB. The tendency of the other component to turn the body about AB is measured by the product of its magnitude $F \sin \theta$ by the perpendicular distance p of its line of action from the axis. Thus the moment of the force F about AB is $Fp \sin \theta$.

We notice that $F \sin \theta$ is the projection of F on a plane perpendicular to the line AB. By projecting the forces on such a plane, we can show that the sum of the moments of two forces about any line is equal to the moment of its resultant about that line. The converse is also true, viz., the moment of a force about a line is equal to the sum of the moments of its components.

10·41. Moments of a force about coordinate axes. Let a force F in space act at the point A whose coordinates are (x, y, z), and let X, Y and Z be the components of the given force F parallel to x-, y- and z-axis respectively.



Then the moment of the force F about the z-axis is equal to the sum of the moments of its components about the z-axis. The moments of X and Y about the z-axis are respectively $-yX$ and xY, while Z has no moment about the z-axis being parallel to it.

Hence the moment M of the force F about the z-axis, is given by

$$M_z = xY - yX.$$

Similarly, the moments of the force F about x-axis and y-axis are respectively

$$M_x = yZ - zY,$$

and $$M_y = zX - xZ.$$

The sign of the moment is taken positive or negative according as the turning effect is anticlockwise or clockwise when viewed from the positive end of the axis towards the origin.

10·42. Couples in space. A couple in space consists of two equal forces acting on a body in opposite senses along two parallel lines. They cause a rotation of the body about an axis perpedicular to the plane containing the two forces. For example, if the couple lies in the yz-plane the axis of rotation is the x-axis. A couple is a vector.

or combined with another couple according to the laws of vectors.

Thus, a couple of moment M with an axis inclined at angles λ, μ, ν to the coordinate axes, can be resolved into three component couples M_x, M_y, M_z whose axes coincide respectively with x-, y- and z-axis and whose magnitudes are given by

$$M_x = M\cos\lambda, \quad M_y = M\cos\mu, \quad M_z = M\cos\nu.$$

Conversely, the couples M_x, M_y, M_z with axes respectively along the three coordinate axes may be combined into a resultant couple M, where

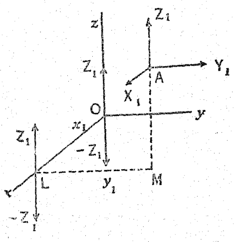
$$M = \sqrt{(M_x^2 + M_y^2 + M_z^2)},$$

and the direction angles of the axis of the resultant couple are given by

$$\cos\lambda = M_x/M, \quad \cos\mu = M_y/M, \quad \cos\nu = M_z/M.$$

10·5. Force system in space. *A system of forces acting on a rigid body can be reduced, in general, to a force acting at a specified point and a couple.*

Let F_1, F_2, ... be a system of forces in space. Take a set of rectangular axes with the specified point as the origin and let (x_1, y_1, z_1), (x_2, y_2, z_2), ... be the coordinates of the points of application of the forces $F_1, F_2, \ldots$ respectively.



Take one of the forces, say F_1, acting at A and let X_1, Y_1, Z_1, be its components parallel to the coordinate axes. Let M be the foot of the perpendicular from A on xy-plane, and let L be the foot of the perpendicular from M on x-axis. Introduce pairs of equal and opposite forces Z_1 and $-Z_1$ each at L and O. These do not affect the resultant of the system.

The force Z_1 at A and $-Z_1$ at L constitute a couple in a plane parallel to yz-plane, of moment y_1Z_1. Similarly the force Z_1 at L and $-Z_1$ at O constitute a couple in the plane xz of moment $-x_1Z_1$.

Hence the force Z_1 at A can be replaced by a force Z_1 acting at O together with two couples of magnitudes y_1Z_1 and $-x_1Z_1$ and axes along Ox and Oy respectively.

In a similar way, the force X_1 acting at A can be replaced by a force X_1 at O together with couples of magnitude z_1X_1 along Oy and $-y_1X_1$ along Oz; and the force Y_1 at A can be replaced by a force Y_1 at O together with couples x_1Y_1 along Oz and $-z_1Y_1$ along Ox.

Hence the force F_1 acting at A is equivalent to the force components X_1, Y_1, Z_1 acting at O, and couples of moment

$$y_1Z_1 - z_1Y_1 \text{ with axis along } Ox,$$
$$z_1X_1 - x_1Z_1 \text{ with axis along } Oy,$$
$$x_1Y_1 - y_1X_1 \text{ with axis along } Oz.$$

The other forces F_2, $F_3, \ldots$ of the system can be treated in the same way, and the system can be reduced to the forces

$$X = \Sigma X_1, \quad Y = \Sigma Y_1, \quad Z = \Sigma Z_1,$$

along the x-, y- and z-axis respectively; and the couples

$$M_x = \Sigma(y_1Z_1 - z_1Y_1), \; M_y = \Sigma(z_1X_1 - x_1Z_1),$$
$$M_z = \Sigma(x_1Y_1 - y_1X_1),$$

along the same three axes.

The forces X, Y, Z can be combined into a single resultant R acting at O, while the couples M_x, M_y, M_z can be combined into a single couple M.

Ex. A system consists of the following forces :

(i) 250 kg. acting at $(3, -5, 8)$ towards the point $(0, -1, 8)$,

(ii) 420 kg. acting at $(-6, 0, 2)$ towards the point $(6, 6, -2)$,

(iii) 300 kg. acting at $(0, 3, -8)$ towards the point $(2, 8, 6)$,

or combined with another couple according to the laws of vectors.

Thus, a couple of moment M with an axis inclined at angles λ, μ, ν to the coordinate axes, can be resolved into three component couples M_x, M_y, M_z whose axes coincide respectively with x-, y- and z-axis and whose magnitudes are given by

$$M_x = M\cos\lambda, \quad M_y = M\cos\mu, \quad M_z = M\cos\nu.$$

Conversely, the couples M_x, M_y, M_z with axes respectively along the three coordinate axes may be combined into a resultant couple M, where

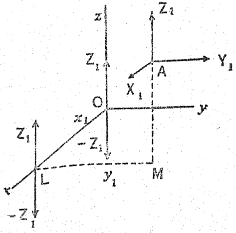
$$M = \sqrt{(M_x^2 + M_y^2 + M_z^2)},$$

and the direction angles of the axis of the resultant couple are given by

$$\cos\lambda = M_x/M, \quad \cos\mu = M_y/M, \quad \cos\nu = M_z/M.$$

10·5. Force system in space. *A system of forces acting on a rigid body can be reduced, in general, to a force acting at a specified point and a couple.*

Let F_1, F_2, ... be a system of forces in space. Take a set of rectangular axes with the specified point as the origin and let (x_1, y_1, z_1), (x_2, y_2, z_2), ... be the coordinates of the points of application of the forces $F_1, F_2, \ldots$ respectively.



Take one of the forces, say F_1, acting at A and let X_1, Y_1, Z_1, be its components parallel to the coordinate axes. Let M be the foot of the perpendicular from A on xy-plane, and let L be the foot of the perpendicular from M on x-axis. Introduce pairs of equal and opposite forces Z_1 and $-Z_1$ each at L and O. These do not affect the resultant of the system.

The force Z_1 at A and $-Z_1$ at L constitute a couple in a plane parallel to yz-plane, of moment $y_1 Z_1$. Similarly the force Z_1 at L and $-Z_1$ at O constitute a couple in the plane xz of moment $-x_1 Z_1$.

Hence the force Z_1 at A can be replaced by a force Z_1 acting at O together with two couples of magnitudes $y_1 Z_1$ and $-x_1 Z_1$ and axes along Ox and Oy respectively.

In a similar way, the force X_1 acting at A can be replaced by a force X_1 at O together with couples of magnitude $z_1 X_1$ along Oy and $-y_1 X_1$ along Oz; and the force Y_1 at A can be replaced by a force Y_1 at O together with couples $x_1 Y_1$ along Oz and $-z_1 Y_1$ along Ox.

Hence the force F_1 acting at A is equivalent to the force components X_1, Y_1, Z_1 acting at O, and couples of moment

$$y_1 Z_1 - z_1 Y_1 \text{ with axis along } Ox,$$
$$z_1 X_1 - x_1 Z_1 \text{ with axis along } Oy,$$
$$x_1 Y_1 - y_1 X_1 \text{ with axis along } Oz.$$

The other forces F_2, F_3, ... of the system can be treated in the same way, and the system can be reduced to the forces

$$X = \Sigma X_1, \quad Y = \Sigma Y_1, \quad Z = \Sigma Z_1,$$

along the x-, y- and z-axis respectively; and the couples

$$M_x = \Sigma(y_1 Z_1 - z_1 Y_1),\ M_y = \Sigma(z_1 X_1 - x_1 Z_1),$$
$$M_z = \Sigma(x_1 Y_1 - y_1 X_1),$$

along the same three axes.

The forces X, Y, Z can be combined into a single resultant R acting at O, while the couples M_x, M_y, M_z can be combined into a single couple M.

Ex. A system consists of the following forces :

(i) 250 kg. acting at (3, -5, 8) towards the point (0, -1, 8),

(ii) 420 kg. acting at (-6, 0, 2) towards the point (6, 6, -2),

(iii) 300 kg. acting at (0, 3, -8) towards the point (2, 8, 6),

(iv) 100 kg. acting at (4, 4, 0) towards the point $(-2, -4, 0)$
(v) 200 kg. along the positive direction of x-axis.

Replace the system of forces by a single force at the origin and a couple. Give the direction of this single force and the direction of the axis of the couple. Find also the angle between them. (The distances are measured in decimetres.)

The solution is given in the table below :—

(See the Table on page 175)

Therefore, the single force R

$$=\sqrt{\{(390)^2+(400)^2+(400)^2\}}=687{\cdot}1 \text{ kg.}$$

$\cos\alpha=390/687=\cdot568$, or $\alpha=55°\ 24'$.
$\cos\beta=400/687=\cdot582$, or $\beta=54°\ 23'$.
$\cos\gamma=-400/687=-\cdot582$, or $\gamma=125°\ 37'$.

The moment M of the couple $=\sqrt{\{(2000)^2+(1520)^2+(1430)^2\}}$
$=2891$ kg.-decimetre $=289{\cdot}1$ kg.-m.

$\cos\lambda=-2000/2891=-\cdot692$, or $\lambda=133°\ 48'$.
$\cos\mu=-1520/2891=-\cdot526$, or $\mu=121°\ 44'$.
$\cos\nu=-1430/2891=-\cdot495$, or $\nu=119°\ 40'$.

The angle θ between the resultant force at the origin and the axis of the couple is given by :

$$\begin{aligned}\cos\theta&=\cos\alpha\cos\lambda+\cos\beta\cos\mu+\cos\gamma\cos\nu\\&=-\cdot568\times\cdot692-\cdot582\times\cdot526+\cdot582\times\cdot495\\&=-\cdot412.\end{aligned}$$

$\therefore\qquad \theta=114°\ 20'$.

10·6. Parallel forces in space. *To find the resultant of a system of parallel forces acting on a body.* Let $Z_1, Z_2, \ldots$ be a system of parallel forces acting at the points $A_1, A_2, \ldots$ of a rigid body. Take a set of rectangular axes Ox, Oy, Oz in which Oz is parallel to the given forces. Let the coordinates of $A_1, A_2, \ldots$ be (x_1, y_1, z_1), (x_2, y_2, z_2), ... Then the only difference in this system and the system considered in § 10·5 is that the X and Y components of the forces are zero. Therefore the give system of parallel forces can be reduced to a single force

$$=\Sigma Z_1$$

acting at O along the z-axis, and couples of moment

$$\Sigma y_1 Z_1 \text{ with axis along } Ox,$$

and $\qquad -\Sigma x_1 Z_1$ with axis along Oy.

Force	from	to	direction cosines	X	Y	Z	$M_x = yZ - zY$	$M_y = zX - xZ$	$M_z = xY - yX$
250 kg.	(3, —5, 8)	(0, —1, 8)	$-\frac{3}{5}, \frac{4}{5}, 0$	—150	200	0	—1600	—1200	—150
420 kg.	(—6, 0, 2)	6, 6, —2)	$\frac{6}{7}, \frac{3}{7}, -\frac{2}{7}$	360	180	—120	—360	0	—1080
300 kg.	(0, 3, —8)	(2, 8, 6)	$\frac{2}{15}, \frac{1}{3}, -\frac{14}{15}$	40	100	—280	—40	—320	—120
100 kg.	(4, 4, 0)	(—2, —4, 0)	$-\frac{3}{5}, -\frac{4}{5}, 0$	—60	—80	0	0	0	—80
200 kg.	(0, 0, 0)	—	1, 0, 0	200	0	0	0	0	0
				390	400	—400	—2000	—1520	—1430

To find if the system reduces to a single force, let us take a force Z acting at the point $(x, y, 0)$. As in § 10·5, this force can be replaced by a force Z at the origin together with a couple yZ along Ox and a couple $-xZ$ along Oy. If this is equivalent to the given system, then

$$Z=\Sigma Z_1,\ yZ=\Sigma y_1 Z_1,\ -xZ=-\Sigma x_1 Z_1.$$

Hence the system of parallel forces has the resultant ΣZ_1 acting at the point $(x, y, 0)$ where

$$x=\frac{\Sigma x_1 Z_1}{\Sigma Z_1},\quad y=\frac{\Sigma y_1 Z_1}{\Sigma Z_1}.$$

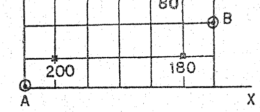
If, however, $\Sigma Z_1=0$, then the system reduces to two couples $\Sigma y_1 Z_1$ along Ox and $-\Sigma x_1 Z_1$ along Oy. These can be combined into a single couple

$$M=\sqrt{\{(\Sigma y_1 Z_1)^2+(\Sigma x_1 Z_1)^2\}}.$$

If M is also zero, the forces are in equilibrium. For M to be zero $\Sigma y_1 Z_1$ and $\Sigma x_1 Z_1$ must separately be zero. Hence *the conditions of equilibrium for a system of parallel forces are*

$$\Sigma Z_1=0,\ \Sigma y_1 Z_1=0 \textit{ and } \Sigma x_1 Z_1=0.$$

Ex. 1. Five weights which are placed on a horizontal table 6 ft. square, have the following magnitudes and coordinates: 100 lb., (3′, 4′); 150 lb., (1′, 5′); 180 lb. (5′, 1′); 80 lb., (4′, 3′) and 200 lb., (1′, 1′) with reference to two adjacent edges as axes. If the table is supported by three legs at the points (0, 0), (6, 2) and (2, 6), find the stresses in the 3 legs.



Referring to the figure we see that we have 8 parallel forces in equilibrium; the cross mark × means forces vertically downwards and the mark ⊙ at the points of support A, B, C, means forces vertically upwards. Take z-axis

perpendicular to the horizontal table, and let R_a, R_b, R_c denote the reactions at A, B, C. Applying the equations of equilibrium, we have,

$$R_a+R_b+R_c-(200+180+80+100+150)=0,$$

or $$R_a+R_b+R_c=710. \quad \ldots \quad (1)$$

$$\Sigma y_1 z_1=2R_b+6R_c-200.1-180.1-80.3-100.4-150.5=0,$$

or $$2R_b+6R_c=1770. \quad \ldots \quad (2)$$

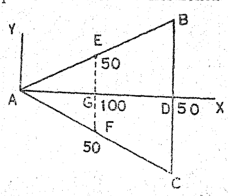
$$\Sigma x_1 z_1=-6R_b-2R_c+200.1+150.1+100.3+80.4+180.5=0,$$

or $$6R_b+2R_c=1870. \quad \ldots \quad (3)$$

From (2) and (3), $R_b=240$ lb. and $R_c=215$ lb.

From (1), $R_a=710-R_b-R_c=710-240-215=255$ lb.

Ex. 2. An equilateral triangular plate ABC is held in horizontal position by means of three vertical cords attached to three angular points A, B and C. A load of 100 lb. is placed at the middle point of the median from A to BC, and three loads of 50 lb. each are placed at the mid-points of AB, BC and CA respectively. Determine the tension in the cords, neglecting the weight of the plate ABC. [*Banaras*, 1963]



Take x and y axes as shown and the z-axis perpendicular to the plate. As the loading is symmetrical about x-axis, the tensions T_b and T_c in the strings at B and C must also be equal.

By replacing the tensions T_b and T_c by their resultant $2T_b$ acting at D, and the load 50 lb. at E and 50 lb. at F by their resultant 100 lb. at G, the given system can be reduced to the following system of coplanar parallel forces:

(i) A tension T_a acting vertically upwards at A,

(ii) A load 200 lb. acting vertically downwards at G, and

(iii) A force $2T_b-50$ acting vertically upwards at D.

Taking moments about the y-axis we get

$$200.AG=(2T_b-50).2AG, \text{ or } T_b=75 \text{ lb.}$$

Resolving vertically, $T_a+(2T_b-50)-200=0$, or $T_a=100$ lb.

10·7. Equilibrium of forces in space. We have seen in § 10·5 that a system of forces in space can be reduced to a single force R and a couple M. For

equilibrium it is necessary that there should be neither a resultant force nor a resultant couple. Therefore, for equilibrium we must have

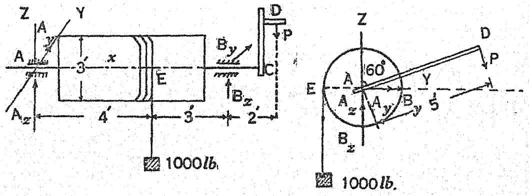
$$R=0 \quad \text{and} \quad M=0.$$

Since $R=\sqrt{(X^2+Y^2+Z^2)}$ and $M=\sqrt{(M_x^2+M_y^2+M_z^2)}$, these can be satisfied only if

$$X=0,\ Y=0,\ Z=0 \text{ and } M_x=0,\ M_y=0,\ M_z=0.$$

This means that for equilibrium the sum of the resolved parts of the forces along three mutually perpendicular lines are separately zero and the sum of the moments of the forces about these three lines are separately zero.

Ex. The figure shows a windlass used in lifting heavy weights. A force P is applied to the crank CD perpendicular to the axis of the cylinder and also perpendicular to the crank CD. Find the value of P required to hold a weight of 1000 lb. and also the reactions at A and B when the crank makes an angle of 60° with the vertical. The bearings at A and B may be assumed to be smooth.



Take a vertical through A as z-axis, the axis of cylinder as x-axis and a perpendicular horizontal line as y-axis. Then the coordinates of the points are :

$A(0, 0, 0)$, $B(7, 0, 0)$, $E(4, -1{\cdot}5, 0)$ and $D\ (9, 2{\cdot}5\sqrt{3}, 2{\cdot}5)$.

The free-body diagram for the windlass gives the following forces in equilibrium.

(i) Reaction of bearing at A, components A_y and A_z.

(ii) Reaction of bearing at B, components B_y and B_z.

(iii) Effort P applied at D.

(iv) The load of 1000 lb. at E.

The components may be tabulated as follows :

Forces	Acting at	direction cosines	X	Y	Z
R_a	(0, 0, 0)	—	0	A_y	A_z
R_b	(7, 0, 0)	—	0	B_y	B_z
1000 lb.	(4, −1·5,0)	0, 0, −1	0	0	−1000
P	(9,4·33,2·5)	0,·5, —·866	0	·5 P	—·866P

Forces	$M_x = yZ - zY$	$M_y = zX - xZ$	$M_z = xY - yX$
R_a	0	0	0
R_b	0	$-7B_z$	$7B_y$
1000 lb.	1500	4000	0
P	$-5P$	$7{\cdot}8P$	$4{\cdot}5P$

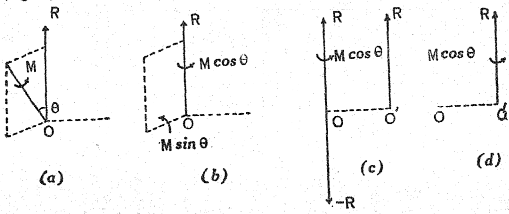
Hence, $M_x=0$ gives $P=300$ lb.
$M_y=0$ gives $B_z=\frac{1}{7}(400+7{\cdot}8P)=905{\cdot}6$ lb.
$M_z=0$ gives $B_y=-\frac{1}{7}(4{\cdot}5P)=-192{\cdot}28$ lb.
$Y=0$ gives $A_y=-B_y-{\cdot}5P=42{\cdot}8$ lb.
$Z=0$ gives $A_z=-B_z+1000+{\cdot}866P=354{\cdot}2$ lb.

Reaction at $A=\sqrt{\{(42{\cdot}8)^2+(354{\cdot}2)^2\}}=357$ lb.
Direction angles are 0, $\cos^{-1}(42{\cdot}8/357)$, $\cos^{-1}(354{\cdot}2/357)$ or 0°,83°,7°
Reaction at $B=\sqrt{\{(192{\cdot}8)^2+(905{\cdot}6)^2\}}=926$ lb.
Direction angles are 0, $\cos^{-1}\{(-192{\cdot}8)/926\}$, $\cos^{-1}(905{\cdot}6/926)$ or 0°, 102°, 12°.

10·8. The wrench. We have seen that a system of forces in space can be reduced to a single force R at a specified point O and a couple of moment M whose axis is, in general, different from the line of action of R. We have also seen that if $R=0$ and $M=0$, the

forces are in equilibrium. If only $R=0$, and $M\neq 0$, the system reduces to a couple of moment M. On the other hand, if $M=0$ and $R\neq 0$, the system reduces to a single force R acting at O.

If neither R nor M is zero, let the axis of the couple M make an angle θ with R, as in fig. (*a*) below. Then the couple M can be resolved into two couples, one of moment $M \cos\theta$ and axis along R, and the other of moment $M\sin\theta$ and axis perpendicular to R (fig. *b*). The latter couple is equivalent to forces R at O' and $-R$ at O where $OO'=(M\sin\theta)/R$, and OO' and R are in a plane perpendicular to the axis of $M\sin\theta$ (fig. *c*). The forces R and $-R$ at O cancel each other;



(*a*) (*b*) (*c*) (*d*)

also the axis of the couple $M\cos\theta$ can be shifted to any parallel axis. So we are left with a single force R acting at O' and a couple $M\cos\theta$ whose axis coincides with the line of action of R (fig. *d*). Such a combination of a force and a couple is called a *wrench*. The line of action of the force is called the axis of the wrench, and the ratio of the moment of the couple $M\cos\theta$ to the magnitude of the force R is called the *pitch* of the wrench.

Ex. Find the resultant wrench of the five forces acting as in ex. on page 174.

Taking the values as obtained in the above mentioned example and resolving the couple M into components parallel and perpendicular to R, we have

$$M\cos\theta=289{\cdot}1\cos 114^\circ 20'=-289{\cdot}1\times{\cdot}412=-119{\cdot}1,$$
$$M\sin\theta=289{\cdot}1\sin 114^\circ 20'=289{\cdot}1\times{\cdot}9112=263{\cdot}4.$$

Hence the system is reduced to a wrench consisting of a force $R=687{\cdot}1$ kg. and a couple of moment $-119{\cdot}1$ kg.-m.. The direction angles of the axis of the wrench are the same as those for R, viz.

$$55^\circ\,24',\ 54^\circ\,23',\ 125^\circ\,37'.$$

10·81. Equation of the axis of a wrench. Let a system of forces be reduced, as in § 10·5, to the force-components

$$X=\Sigma X_1,\ Y=\Sigma Y_1,\ Z=\Sigma Z_1,$$

acting along the three axes of coordinates and to the couples

$$M_x=\Sigma(y_1Z_1-z_1Y_1),\ M_y=\Sigma(z_1X_1-x_1Z_1),\ M_z=\Sigma(x_1Y_1-y_1X_1)$$

along the same three axes.

Let us shift the origin from O to any other point O' whose coordinates are $(\xi,\ \eta,\ \zeta)$ and take a set of parallel axes there. Then the force-components in the new set of axes are still $X,\ Y,\ Z$, but the couples along the three axes change to $M'_x,\ M'_y,\ M'_z$, where

$$\begin{aligned}M'_x&=\Sigma\{(y_1-\eta)Z_1-(z_1-\zeta)Y_1\}\\&=\Sigma(y_1Z_1-z_1Y_1)-\eta\Sigma Z_1+\zeta\Sigma Y_1\\&=M_x-\eta Z+\zeta Y,\end{aligned}$$

and similarly

$$M'_y=M_y-\zeta X+\xi Z,\ M'_z=M_z-\xi Y+\eta X.$$

If the point O' lies on the resultant wrench of the system then the line of action of R and the axis of the resultant couple M' have the same direction cosines, i.e.

$$\frac{X}{R}=\frac{M'_x}{M'},\ \frac{Y}{R}=\frac{M'_y}{M'},\ \frac{Z}{R}=\frac{M'_z}{M'},$$

or

$$\frac{M'_x}{X}=\frac{M'_y}{Y}=\frac{M'_z}{Z}=\frac{M'}{R},$$

or

$$\frac{M_x-\eta Z+\zeta Y}{X}=\frac{M_y-\zeta X+\xi Z}{Y}=\frac{M_z-\xi Y+\eta X}{Z}.$$

Hence $(\xi,\ \eta,\ \zeta)$ lies on the line

$$\frac{M_x-yZ+zY}{X}=\frac{M_y-zX+xZ}{Y}=\frac{M_z-xY+yX}{Z}.$$

This is the equation of the axis of the wrench. It is also known as *Poinsot's central axis.*

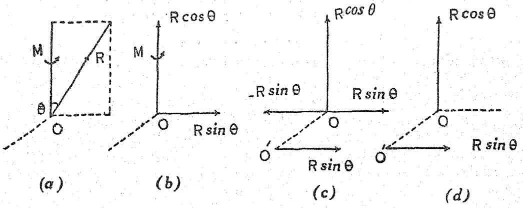
10·82. Resolution into two non-intersecting forces. *A given system of forces can, in general, be reduced to two non-intersecting forces.*

We have seen in § 10·5 that a given system of forces can be reduced to a single force R at O and a couple M. Let R make an angle θ with the axis of the couple (fig. a). Then R can be resolved

into a component $R \cos\theta$ along the axis of M and $R \sin\theta$ perpendicular to M (fig. b). The couple M is equivalent to a force

$$-R \sin\theta \text{ at } O \text{ and } R \sin\theta \text{ at } O',$$

where $OO' = M/R \sin\theta$, and OO' and $R \sin\theta$ are in a plane perpendicular to the axis of M (fig. c).



The forces $R \sin\theta$ and $-R \sin\theta$ at O cancel each other, and we are left with the two non-intersecting forces $R \cos\theta$ at O and $R \sin\theta$ at O' (fig. d).

EXAMPLES 17

1. Four couples of moments 65 kg.-m., 70 kg.-m., 75 kg.-m. and 85 kg.-m., act in the planes in space whose normals have direction cosines proportional to $3:4:12, -6:3:2, 2:5:-14$ and $8:15:0$ respectively. Find the resultant couple and the direction cosines of its axis.

2. A force of 910 lb. acts at the point (2′, 3′, 4′) with the direction and sense of the line segment joining this point to (5′, 7′, 16′); a force of 750 lb. acts at point (−2′, −1′, 2′) with the direction and sense of the line segment joining this point to (−9′, −25′, 2′); a force of 440 lb. acts at origin in the positive sense of y-axis and a force of 840 lb. acts at origin in the negative sense of z-axis. Replace this system of forces by a single force acting at the origin and a couple. [*Banaras*, 1957]

3. Forces, perpendicular to xy-plane, act at points as given below :

40 kg.	upwards	through	(1, −3),
50 kg.	downwards	through	(−2, 4),
20 kg.	upwards	through	(8, −6),
60 kg.	upwards	through	(4, 0),
30 kg.	downwards	through	(3, −8) and
10 kg.	downwards	through	(0, 7).

Find the resultant of the system and its position.

4. An equilateral triangular plate ABC of weight 40 kg. is he d in a horizontal position by means of three vertical cords

attached from the three angular points A, B and C to a ceiling. A load of 50 kg. is placed at the mid-point of the perpendicular from A to BC and two loads of 25 kg. each, are placed at the mid-points of AB and AC. Find the tensions in the three cords.

5. A square plate of weight W is supported in a horizontal position by three legs. The two legs at the adjacent corners carry one-fourth and one-fifth of the weight respectively. Find the position of the point on the plate where the third leg meets it. [*Banaras*, 1953]

6. A tricycle of weight 40 kg. has a small wheel symmetrically placed 1 metre in the front of the line joining the two large wheels which are 90 cm. apart. If the centre of gravity of the machine be 30 cm. in front of the hind wheels and a rider whose weight is 75 kg. be 20 cm. in front of the hind wheels, find how the weight is distributed on different wheels.

7. A circular table of weight 50 kg. is supported by three legs equally spaced on the circumference. A weight of 40 kg. is placed on the table midway between the centre and one leg; a weight of 30 kg. is placed on the table one-fourth the way from centre to another leg. Find the force of compression in each leg of the table.

8. A circular table, 6 feet in diameter, weighing 128 pounds, rests on three legs equally spaced and each 2 feet from the centre. Find the least weight which, when hung at the edge of the table, will make tipping impending. (Assume the weight of the table to be concentrated at the centre.) [*Banaras*, 1955]

9. Two men hoist a stone of weight 600 lb. at a constant speed out of a pit by means of a rope that passes from the stone over a pulley directly above it and is then wound upon a windlass 16 inches in diameter. If the cranks on the windlass are 2 feet long, find the force each man must exert perpendicular to the cranks. [*Agra*, 1957]

10. A force of 50 pounds acts from the point $(0, 0, 0)$ towards the point $(3, 4, 0)$; a force of 120 pounds from $(1, 2, 0)$ towards $(1, 2, -4)$; a force of 130 pounds from $(2, 3, 4)$ towards $(5, 7, 16)$. Find the resultant of the system.

Replace the system by an equivalent wrench and determine its pitch. [*Banaras*, 1952]

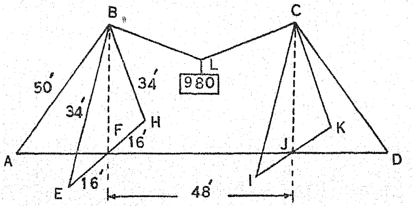
11. Forces P, Q, R, P, Q, R act along the edges BC, CA, AB, AD, BD, CD of a regular tetrahedron $ABCD$. Show that they are equivalent to a wrench of pitch $a/2\sqrt{2}$, where a is the length of an edge.

[Hint. Take the six edges of the tetrahedron as six diagonals of the faces of a cube of side $a/\sqrt{2}$.]

12. A square trap-door $ABCD$, capable of turning about smooth hinges at A and B, in a horizontal line, is held at an angle of 60° with the vertical by a rope attached at C to a point E vertically above B. If the weight of the door is 120 kg. and $BE=BC=2a$, find the tension of the rope and the reactions at the hinges A and B.

13. A trap-door, 6 feet by 4 feet, weighing 100 pounds, has hinges A and B, one foot from each end of a horizontal 6-foot edge. It is held in a horizontal position by means of a cord from the mid-point of the free 6-foot edge to a point 4 feet vertically above the hinge A. Find the reactions on the hinges and the tension in the cord if the hinges are loose so that only the hinge A has a component of reaction parallel to the hinged edge of the door. [*Banaras*, 1962]

14. The given figure represents a lifting device. A load of 980 pounds is suspended from a smooth ring on cable BLC and produces a sag of 7 feet below the level of BC. Find the forces in BC, in the tie AB, and in the stay BE. [*Banaras*, 1963]



15. A rod AB of length l and weight W is pivoted smoothly at one end A and the other end B rests against a vertical wall at a distance a from the pivot A. If the coefficient of friction between the rod and the wall is μ, and the rod is in limiting equilibrium find the angle which the plane AOB makes with the vertical, O being the foot of the perpendicular from A on the wall.

16. A heavy rod is suspended from two equal vertical strings of length l from two points in the same horizontal plane. Show that the magnitude of the couple required to keep it turned through an angle ϕ in a horizontal plane is

$$W a^2 \sin\phi/\sqrt{(l^2-4a^2\sin^2\tfrac{1}{2}\phi)},$$

where $2a$ is the length and W the weight of the rod.

Show also that the tension in the strings is changed in the ratio

$$l:\sqrt{(l^2-4a^2\sin^2\tfrac{1}{2}\phi)}.$$

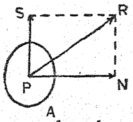
HYDROSTATICS

CHAPTER XI

PRESSURE OF HEAVY FLUIDS

11·1. Introduction. Hydrostatics is that branch of physical science which deals with the conditions under which fluid masses remain in a state of relative equilibrium when acted by a system of forces. While statics and dynamics deal with the action of forces on a solid body, hydrostatics is concerned with the equilibrium of fluids (i.e. liquids and gases).

A fluid is a substance which flows or is capable of flowing. To arrive at a rigorous definition of a fluid, consider a point P within a material and imagine a small plane surface A drawn at P inside the material. Then the two parts into which the material is divided by the surface A, exert equal and opposite forces R on each other. (Only one of these forces is shown in the figure above.)



The force R can be resolved into two components N and S, one normal to the surface A and the other lying in it. The component N along the normal is called the normal force, and the force S in the plane A is called the shearing force. The shearing force per unit area is called the shearing stress.

The particles of a solid body are linked to each other in such a fashion that when shearing forces are applied to the body, the particles resist motion relative to each other. The particles (or molecules) of a liquid or a gas are not linked to each other in this fashion; and when forces are applied which create shearing stresses inside the material, the particles slide over one another and relative motion ensues. So we say that *a fluid is*

a substance which yields to any continued shearing stress however small.

This definition applies to substances differing as widely as air, water, molasses and coal tar. The difference between these substances can be explained by observing that they exert resistances of different magnitudes when their particles slide over each other. This property of the fluids is known as viscosity, and is of the nature of friction. The greater the resistance to relative motion offered by a fluid, the larger is said to be its viscosity. The resistance appears as a shearing stress opposing the relative motion.

When shearing forces act on a fluid mass, motion ensues. Due to this motion shearing stresses opposing the motion develop inside the fluid on account of viscosity. These dissipate the motion unless the external shearing forces continue to act. When there is no motion in the fluid, there are no shearing stresses inside it. For if there were a shearing stress, motion will ensue. Hence *there are no shearing stresses in a fluid in equilibrium.*

A *perfect fluid* is an ideal substance which is incapable of exerting a shearing stress, whether in motion or in equilibrium. Thus the force on a plane surface in contact with a perfect fluid is always normal to the surface. Moreover, no resistance will be offered by a perfect fluid to the relative motion of its particles.

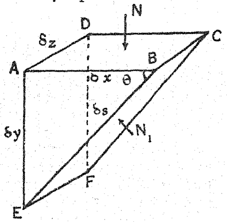
Fluids are of two kinds : liquids and gases. A liquid (e.g. water or oil) is incompressible or nearly so. Its volume changes very little by an application of pressure. A gas (e.g. air) is easily compressible and changes its volume when the pressure applied to it is changed.

11·2. Pressure at a point. We have used above the term 'pressure' without defining it. As its name indicates, pressure means the force or thrust with which a surface is being pressed. Its measure is the force acting per unit area of the surface.

Let us take a plane area A immersed in a fluid and surrounding a point on it. We have seen that when the fluid is in equilibrium, shearing forces are absent and the forces acting on the plane area are normal to it. Let N be the resultant of these normal forces. Then N/A, the resultant normal force acting on a plane are a immersed in a fluid divided by the area, is called the mean pressure on the area. Let us successively take smaller and smaller plane areas surrounding a point P inside the fluid. Let A be one of these areas and N the resultant normal force acting on it, then the limiting value of N/A, when the area A tends to zero, is called the *pressure at the point P*.

We shall now show that the pressure at a point does not depend on the direction in which the plane area A is taken. In other words, *the pressure at a point in a fluid in equilibrium is the same in every direction.*

PROOF. Consider an elementary portion of the fluid in the form of a right triangular prism $ABCDEF$, where $AB=\delta x$, $AE=\delta y$, $AD=\delta z$, $BE=\delta s$, and $\angle ABE=\theta$, as in the figure.



Let N and N_1 be the resultant normal forces on the faces $ABCD$ and $BCEF$, and let p and p_1 be the pressures on them respectively; then

$$\left.\begin{aligned} N &= p\,\delta x\,\delta z, \\ \text{and} \quad N_1 &= p_1\,\delta s\,\delta z = p_1\,\delta x \delta z/\cos\theta. \end{aligned}\right\} \tag{1}$$

The resultant forces on all the other faces of the prism are in directions which are perpendicular to N. Therefore resolving the forces parallel to N, we have

$$N + Y \,.\, \tfrac{1}{2}\delta x\,\delta y\,\delta z = N_1 \cos\theta, \tag{2}$$

where Y is the component of body forces (e. g. weight)

parallel to N, per unit volume, acting on the fluid. Substituting in (2) from (1), and dividing by $\delta x\,\delta z$, we get

$$p+\tfrac{1}{2}Y\,\delta y=p_1.$$

If δx, δy and δz tend to zero, so that the prism shrinks to a point, we obtain from the above equation that

$$p=p_1.$$

Since the directions of the axes and the value of θ can be chosen arbitrarily, we see that the pressure at a point is the same in all directions.

11·3. Density. The density of a substance is its mass per unit volume. Thus if M be the mass of a volume V of a substance, its density ρ is equal to M/V, or

$$M=\rho V.$$

The units for density are gram per cubic centimetre in the metric system and pounds per cubic foot in f.p.s. system.

If W denotes the weight in absolute units of a mass M of a substance, then as in § 3·21, part II,

$$W=Mg$$
$$=\rho gV,$$

where g is the acceleration due to gravity.

Putting $V=1$ in the above formula, we see that the weight w of unit volume of a fluid of density ρ is given by

$$w=\rho g.$$

The *specific gravity* of a substance is the ratio of the density of the substance to the density of a standard substance. The substance usually adopted as the standard is pure water at the temperature of 4°C (when its density is the maximum). In metric system this density is 1 gram per cubic centimetre. In f.p.s. system it is 62·425, or roughly 62·5 pounds per cubic foot.

Ex. Find the specific gravity of a mixture of three liquids of volumes V_1, V_2, V_3, and specific gravities s_1, s_2, s_3.

Let ρ be the density of the standard substance. Then the densities of the three liquids are $s_1\rho$, $s_2\rho$, $s_3\rho$; and their masses are $V_1s_1\rho$, $V_2s_2\rho$, $V_3s_3\rho$. Therefore the mass of the mixture

$$=V_1s_1\rho+V_2s_2\rho+V_3s_3\rho,$$

and its volume is $V_1+V_2+V_3$.

Hence the density of the mixture is

$$(V_1s_1+V_2s_2+V_3s_3)\rho\div(V_1+V_2+V_3),$$

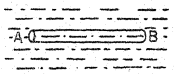
and its specific gravity is

$$(V_1s_1+V_2s_2+V_3s_3)/(V_1+V_2+V_3).$$

11·4. Pressure in a heavy fluid. We shall now establish two theorems for determining the pressure inside a fluid which is in equilibrium under the action of gravity. In this case the only external force acting on the fluid is its weight.

(i) *In a fluid in equilibrium under gravity, the pressures at any two points in the same horizontal plane are equal.*

Let A, B be any two points in the same horizontal plane and let the straight line AB lie wholly in the fluid. Describe a thin cylinder with AB as axis and plane ends normal to AB. Consider the equilibrium of the fluid contained in this cylinder and resolve the forces acting on it in the direction AB. The only forces having a component in direction AB are the forces on the plane ends. Hence the forces on the two plane ends are equal. But the areas of the two plane ends are equal. So the mean pressures on the two plane ends are also equal. Taking thinner and thinner cylinders, we see, therefore, that the pressure at A is equal to the pressure at B.



(ii) *In a homogeneous fluid in equilibrium under gravity the difference between pressures at two points is equal to the difference between their depths multiplied by the weight of unit volume of the fluid.*

Let A, B be two points inside the fluid in the same vertical line at distance d from each other. Describe a thin cylinder of cross section α with AB as axis and horizontal plane ends at A and B. Let p and p' be

parallel to N, per unit volume, acting on the fluid. Substituting in (2) from (1), and dividing by $\delta x\,\delta z$, we get

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$$=V_1s_1\rho+V_2s_2\rho+V_3s_3\rho,$$

and its volume is $V_1+V_2+V_3$.

Hence the density of the mixture is

$$(V_1s_1+V_2s_2+V_3s_3)\rho\div(V_1+V_2+V_3),$$

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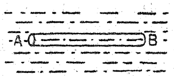
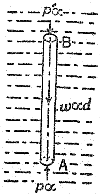


(ii) *In a homogeneous fluid in equilibrium under gravity the difference between pressures at two points is equal to the difference between their depths multiplied by the weight of unit volume of the fluid.*

Let A, B be two points inside the fluid in the same vertical line at distance d from each other. Describe a thin cylinder of cross section α with AB as axis and horizontal plane ends at A and B. Let p and p' be

the pressures at A and B and let w be the weight of unit volume of the fluid.

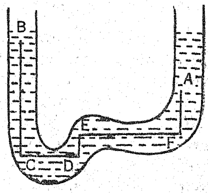


Consider the equilibrium of the fluid contained in the cylinder. The vertical forces acting on it are the forces $p\alpha, p'\alpha$ on the plane ends at A and B, and the weight $w\alpha d$ of the cylinder. Equating these, we get

$$p'\alpha + w\alpha d = p\alpha,$$

or
$$p - p' = wd.$$

When A and B are not in the same vertical line we can take a point C vertically below B and in the same horizontal plane as A. Then by (i), the pressure at A is same as the pressure at C. Also the difference between pressures at C and B is equal to wd, where $d = BC$.



It is easily seen that the theorem is true even in cases where a more complicated set of horizontal and vertical lines are needed to join A and B as in the marginal diagram.

11·5. Pressure at depth d. The above theorem shows that the pressure at a point inside a liquid of weight w per unit volume at a depth d below the surface is

$$\mathbf{wd}$$

on account of the weight of the liquid. This may also be written as

$$\rho \mathbf{gd},$$

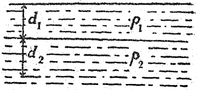
where ρ is the density of the liquid and g the acceleration due to gravity.

In general, the liquid surface is in contact with atmosphere which exerts a pressure due to the weight of the air. This pressure, which is approximately equal

to the weight of a column of mercury 76 cm. high, varies slightly from day to day and also with the height above sea level. For practical purposes it is often taken as constant, and denoted by Π. Thus the pressure at the surface of the liquid is Π due to atmospheric pressure, and the total pressure at a depth d below the surface of the liquid is

$$\Pi + wd, \text{ or } \Pi + \rho \mathbf{g}\mathbf{d}.$$

Similarly, the pressure at a point below a depth d_1 of a liquid of density ρ_1 and depth d_2 of another liquid of density ρ_2, is



$$\Pi + \rho_1 g d_1 + \rho_2 g d_2.$$

The pressure of the atmosphere is often neglected in calculating the pressure at a point. In the following we shall consider the pressure due to the liquid alone unless mentioned specifically.

It is easily seen from the above that the surface of a heavy liquid is horizontal. For, if A and B are any two points in the liquid in the same horizontal plane, and C and D are two points in the surface respectively above A and B, then

$$\text{the pressure at } A = \text{the pressure at } B,$$

or $$w.AC = w.BD,$$

or $$AC = BD,$$

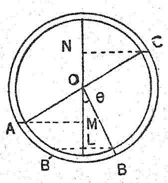
showing that C and D are also in one horizontal plane.

For a similar reason, the common surface of separation of two liquids is also horizontal.

Ex. 1. A uniform tube of small cross section is bent into the form of a circle, whose plane is vertical. Equal quantities of fluids, of densities ρ and σ, fill half of the tube. Show that the radius passing through the common surface makes with the vertical an angle

$$\tan^{-1} \frac{\rho - \sigma}{\rho + \sigma}.$$

Let AB and BC be the fluids of densities ρ and σ, each filling a quarter of the tube. Let $\rho > \sigma$, so that AB rests lower than BC. Let the radius OB make an angle θ with the vertical. Draw the horizontal lines AM, BLB' and CN.



Now B and B' are two points in one fluid in the same horizontal plane. So the pressures at B and B' are equal. But the pressure at B is due to the column BC of liquid σ and that at B' due to the column AB' of liquid ρ. Therefore

$$NL.\sigma g = ML.\rho g,$$

or $$(a \sin\theta + a\cos\theta)\sigma g = (a\cos\theta - a\sin\theta)\rho g,$$

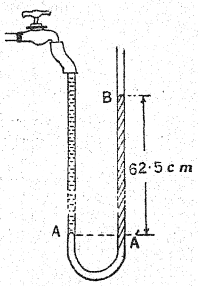
where a is the radius of the circle. This gives

$$(\rho+\sigma)\sin\theta = (\rho-\sigma)\cos\theta,$$

or $$\theta = \tan^{-1}\frac{\rho-\sigma}{\rho+\sigma}.$$

Ex. 2. In order to determine the pressure of a water supply, the main water tap is connected by means of thick tubing to one arm of a U-tube containing mercury. If the level of mercury in the second arm rises 62·5 cm. above the level in the first, find the height of the water level of the supply above the latter level. What is the pressure of the water at this level?

Let A be the surface of mercury in the first arm of the U-tube and B the surface in the second arm. Let A' be the point in the second arm at the same level as A. Then the pressures at A and A' are equal. The pressure at A is due to the water supply. The pressure at A' is due to the column $A'B$ of mercury. Therefore the pressure of the water supply at A



$= \rho g h$

$= 13{\cdot}6 \times 62{\cdot}5$ grams weight per sq. cm.

$= 850$ grams weight per sq. cm.

If h_1 is the height of the water level of the supply above A, the

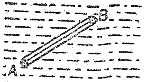
$$1 \,.\, g h_1 = \rho g h,$$

or $$h_1 = 13{\cdot}6 \times 62{\cdot}5 \text{ cm.}$$
$$= 8{\cdot}5 \text{ metres.}$$

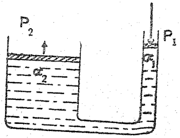
NOTE. A U-tube containing a liquid of known density used for measuring the pressure of water or gas supply, is called a *manometer*.

11·6. Transmissibility of liquid pressure. *If the pressure at any point of a liquid in equilibrium is increased, the increase is transmitted to every other point of the liquid.*

Let A and B be any two points in the liquid. Describe a thin cylinder with AB as axis and plane ends of area a normal to AB. The forces in the direction AB are the pressures on the plane ends and the components of external forces. These balance together. If the pressure at A is now increased by p, an additional force pa will act in the direction AB. To keep the cylinder in equilibrium an additional force pa must act at B in the opposite direction BA. Thus the pressure at B is also increased by the same amount p.



This principle is used in the apparatus known as *hydraulic* or *Bramah's press*. It consists of two cylinders filled with water with a tube at the base connecting each other. One of the cylinders is of small diameter and the other has a large diameter, and both are fitted with tight pistons.



If a force P_1 is applied to the smaller piston of area a_1, the pressure developed is P_1/a_1. This is transmitted to the larger piston of area a_2, and the force P_2 developed there

$$= (P_1/a_1)\,a_2.$$

By increasing the ratio a_2/a_1 a very large force can be developed. There are, however, practical limitations to the force P_2 due to the leakage of water and the strength of the cylindrical vessels.

Ex. In a hydraulic press the radii of the cylinders are 8 centimetres and 2 metres respectively. The power is applied at the end of a lever 60 cm. long, and the piston is attached at a distance of 5 cm. from the fulcrum. If a body weighing 10 metric tonnes be placed upon the larger piston, find the force that must be applied to the lever.

Let P be the force applied to the lever and P_1 be the force on the smaller piston. Then taking moments about the fulcrum,

$$P \,.\, 60 = P_1 \,.\, 5,$$

or

$$P = \tfrac{1}{12} P_1.$$

Now, the pressure on the smaller piston

$$= P_1/\pi \,.\, 8^2,$$

and the pressure on the larger piston

$$= 10 \times 1000/\pi \,.\, 200^2 \text{ kg. per sq. cm.}$$

Therefore

$$\frac{P_1}{\pi \,.\, 8^2} = \frac{10000}{\pi \,.\, 200^2},$$

or

$$P_1 = \frac{64 \times 10000}{40000} = 16 \text{ kg.}$$

Hence

$$P = \tfrac{1}{12} P_1 = 1\tfrac{1}{3} \text{ kg.}$$

Examples 18

1. Calculate the pressure due to water at a depth of one kilometre below the surface of a lake.

2. At what depth below the surface of the sea is the pressure due to the water $\frac{1}{2}$ metric tonne per sq. cm. Specific gravity of sea water is $1{\cdot}03$.

3. The pressure in the water pipe at the basement of a building is $2{\cdot}7$ kg. per sq. cm. and at the third floor it is $1{\cdot}4$ kg. per sq. cm. Find the height of this floor above the basement.

4. The pressure at the bottom of a well is four times that at a depth of 2 feet. What is the depth of the well if the pressure of the atmosphere is equivalent to that of 30 feet of water?

5. A vessel whose bottom is horizontal contains 3 inches of mercury and water on the top of that to a depth of 24 inches. Find the pressure at a point on the bottom of the vessel due to the liquids in lb. wt. per sq. inch, given that 1 cu. ft. of water weighs $62{\cdot}5$ lb. and the specific gravity of mercury is $13{\cdot}6$.

6. A layer of oil 25 cm. deep rests on water contained in a vessel. Calculate the pressure in dynes per sq. cm. at a point 30 cm. below the surface of the water, if the pressure of the atmosphere is equal to that of a mercury column 76 cm. high. (Specific gravity of oil $= 0{\cdot}92$, of mercury $= 13{\cdot}6$, $g = 981$ cm./sec.)

7. In the lower half of a uniform circular tube one quadrant is occupied by a liquid of density 2ρ and the other by two liquids, which do not mix, of densities 3ρ and ρ. Prove that the volume of the lower of the two latter liquids is twice that of the other.

8. A closed tube in the form of an equilateral triangle contains equal volumes of three liquids which do not mix, and is placed with its lowest side horizontal. Prove that, if the densities of the liquids are in arithmetical progression, their surface of separation will be at points of trisection of the sides of the triangle.

9. The lower ends of two vertical tubes of diameters 2 cm. and 0·8 cm. respectively are connected by a tube. The tubes contain mercury of specific gravity 13·6. If 132 c.c. of water are poured into the larger tube, by how much is the level of mercury in the smaller tube raised?

10. The difference in the levels of water in the two limbs of a manometer connected to a gas supply is 15·6 cm. Calculate the pressure of the gas in excess of atmospheric pressure in dynes per sq. cm.

11. A U-tube contains some mercury in the bend, and alcohol is poured into one arm to a depth of 20 cm. What depth of water must be poured into the other arm to bring the two mercury surfaces to the same level ? What depth of water is required if the upper surfaces of the water and the alcohol are to be at the same level? Specific gravity of alcohol is 0·8, and that of mercury 13·6.

12. In a hydraulic press the diameters of the large and small pistons are respectively 25 cm. and 2 cm. A kilogram is placed on the top of the small piston; find the mass it will support on the large piston.

13. In a hydraulic press the area of the larger piston is 600 sq. cm. and that of the smaller one 1·5 sq. cm. Find the force that must be applied to the smaller piston so that the larger one may lift 1 metric tonne.

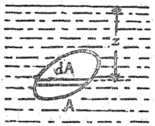
14. In a Bramah's press if a force of 1 ton wt. be produced by an effort equal to 5 lb. wt. and the diameters of the pistons be in the ratio 8 : 1, find the ratio of the lengths of the arms of the lever employed to work the piston.

15. A hydraulic lift (on the principle of Bramah's press) consists of a ram of cross-section A and a piston of cross-section B. The ram supports a weight W, and the piston is driven into the cistern with a slow uniform motion by an engine of power H. The friction at the collar of the ram is F, and at the collar of the piston is F'. Prove that the speed at which W rises is

$$BH/(BW+BF+AF').$$

11·7. Thrust on a plane area. *To find the thrust on a plane area immersed in a heavy homogeneous liquid.*

Let A be a plane area immersed in a heavy liquid whose weight per unit volume is w. Consider an elementary horizontal strip dA of this area whose depth below the surface of the liquid is z. Then the pressure due to the liquid at any point of this element is wz. Therefore, the thrust of the liquid on the element is



$$wz\,dA.$$

Summing up over all the elements, we see that the total thrust F on the area is given by

$$F=\int wz\,dA. \quad . \quad . \quad . \quad (1)$$

If $\bar{z}$ be the depth of the centre of gravity of the area A below the surface of the liquid, we know that

$$\bar{z}=\frac{\int z dA}{A}.$$

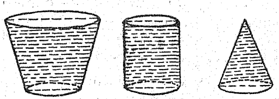
Therefore

$$\int z dA=\bar{z}A.$$

Substituting in (1), we get

$$\mathbf{F}=\mathbf{w\bar{z}A}.$$

The student should note that the thrust on a plane surface depends only on its area and the depth of its centre of gravity. It does not depend on the amount of liquid present in the vessel. For example, if we take three vessels as in the figure, of same height and on equal bases, and fill them with the same liquid, the thrusts on the bases will be equal even though the liquid contained in the vessels are different. The reason is that a part of the weight of the liquid in the first vessel is supported by the sides of the vessel; while in the third vessel the curved surface presses the liquid down (see § 12·1).



Ex. 1. Find the thrust on a triangular area of height h and base b, immersed vertically in a heavy liquid with the base in the surface of the liquid.

Here $A=\frac{1}{2}bh$, and $z=\frac{1}{3}h$.

Therefore the thrust

$$=\tfrac{1}{6}bh^2w,$$

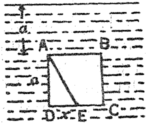
where w is the weight of unit volume of the liquid.

Ex. 2. A square $ABCD$ of side a is immersed in water with its plane vertical and the upper side AB at a depth a below the surface. Show how to draw a line through A which divides the square in two parts the thrusts on which are equal.

Let the required line be AE, cutting CD in E at distance x from D. Then, the thrust on ADE

$$=\text{thrust on } ABCE$$

$$=\tfrac{1}{2}(\text{thrust on } ABCD).$$



Now, the thrust on ADE

$$=\tfrac{1}{2}ax(a+\tfrac{2}{3}a)w=\tfrac{5}{6}a^2xw.$$

The thrust on $ABCD=a^2(a+\frac{1}{2}a)w=\frac{3}{2}a^3w$.

Therefore

$$\tfrac{5}{6}a^2xw=\tfrac{1}{2}(\tfrac{3}{2}a^3w),$$

or

$$x=\tfrac{9}{10}a.$$

11·8. Centre of Pressure. When a plane area is immersed in a fluid, the force exerted by the fluid on each element of area acts in a direction perpendicular to the plane of the area. The magnitudes of the forces on the various elements of area vary, depending upon the depths of these elements below the surface, but the directions of the forces are the same. Hence these forces constitute a system of parallel forces which can be compounded into a single force. The point on the plane area through which the resultant of these parallel forces acts, is called the *centre of pressure* of the area.

To find the depth of the centre of pressure of a plane area immersed in a heavy homogeneous liquid.

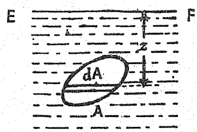
Let A be the area immersed in a liquid of weight w per unit volume (see figure below). Let dA be an element of this area at a depth z below the surface of the liquid. The thrust on this area is $wzdA$ acting at a depth z. Applying the formula for parallel forces (§ 10·6, art I), the depth at which the resultant thrust acts is given by

$$\bar{z}_p = \frac{\Sigma wz\,dA.\,z}{\Sigma wz\,dA},$$

or, replacing the summation by integration,

$$\bar{z}_p = \frac{\int z^2 dA}{\int z dA}. \quad . \quad . \quad . \quad (1)$$

If the area A is in a vertical plane which cuts the surface of the liquid in EF, the depth z is the distance of the element from EF. In this case, the denominator $\int z dA$ is equal to Ah, where h is the distance of the centre of gravity of A from EF; and the numerator $\int z^2 dA$ is equal to Ak^2, where k is the radius of gyration of the area A about EF. Thus, (1) becomes



$$\bar{z}_p = k^2/h. \quad . \quad . \quad . \quad (2)$$

By the theorem of parallel axes (Part II, § 8·5),

$$k^2 = k_G{}^2 + h^2,$$

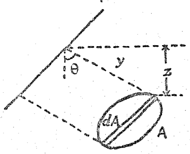
where k_G is the radius of gyration of A about an axis parallel to EF through the centre of gravity G. Substituting in (2), this gives

$$\bar{z}_p = \frac{k_G{}^2}{h} + h, \quad . \quad . \quad . \quad (3)$$

that is, the depth of the centre of pressure below G is $k_G{}^2/h$.

If the area A is not in a vertical plane, but in a plane inclined at an angle θ to it, let EF be the line of intersection of the plane with the surface of the liquid. Let y denote the distance of dA from EF, then

$$z = y \cos \theta.$$



Substitution in (1) shows that the distance $\bar{y}_p$ of the centre of pressure from EF, is given by

$$\bar{y}_p = \frac{\int y^2\, dA}{\int y\, dA} = \frac{k^2}{h},$$

where k is the radius of gyration of A about EF, and h is the distance of the centre of gravity of A from EF.

Ex. 1. Find the depth of the centre of pressure of a rectangle of sides a and b, just immersed vertically in a heavy liquid with one side of length a in the surface.

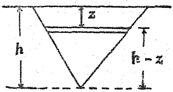
Here, the radius of gyration k about the side a is given by

$$Ak^2 = \tfrac{1}{3}Ab^2.$$

Also the depth h of centre of gravity below the surface is $\tfrac{1}{2}b$.

Therefore $\bar{z}_p = k^2/h = (\tfrac{1}{3}b^2)/(\tfrac{1}{2}b) = \tfrac{2}{3}b.$

Ex. 2. Find the depth of the centre of pressure of a triangle of height h and base b, just immersed vertically in a heavy liquid with (i) base in the surface, (ii) vertex in the surface and base horizontal.



(i) If we take a horizontal strip of the triangle as our element of area, we see from the figure that its length $= k(h-z)$, where k is some constant. Therefore

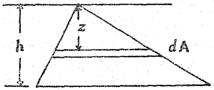
$$\bar{z}_p = \frac{\int z^2\, dA}{\int z\, dA} = \frac{\int_0^h z^2 k(h-z)\, dz}{\int_0^h zk(h-z)\, dz} = \frac{\left[\tfrac{1}{3}hz^3 - \tfrac{1}{4}z^4\right]_0^h}{\left[\tfrac{1}{2}hz^2 - \tfrac{1}{3}z^3\right]_0^h}$$

$$= \frac{\tfrac{1}{12}h^4}{\tfrac{1}{6}h^3} = \tfrac{1}{2}h.$$

(ii) In this case $dA = kz\, dz$.

Therefore

$$\bar{z}_p = \frac{\int_0^h kz^3\, dz}{\int_0^h kz^2\, dz} = \frac{\tfrac{1}{4}kh^4}{\tfrac{1}{3}kh^3} = \tfrac{3}{4}h.$$



NOTE. The above results can be used as standard formulae.

Ex. 3. A water reservoir has a circular hole of radius a in its side which is closed by a circular plate hinged at the top and fastened by a catch at the bottom. Find the force exerted by the catch when the height of the water surface is h above the centre of the plate.

The total thrust on the circular plate

$$=wh \,.\, \pi a^2.$$

The depth of centre of pressure below the surface of water

$$=h+kG^2/h$$
$$=h+(\tfrac{1}{4}a^2)/h.$$

Thus the thrust due to water is $\pi a^2 hw$ acting at a point $a^2/4h$ below the centre of the plate.

If F is the force exerted by the catch on the plate, we get, on taking moments about the hinge,

$$\pi a^2 hw\left(a+\frac{a^2}{4h}\right)=F \,.\, 2a,$$

or

$$F=\tfrac{1}{8}\pi a^2(4h+a)w.$$

EXAMPLES 19

1. A cube of 30 cm. edge is immersed in water with its upper face horizontal and at a depth 75 cm. below the surface. Find the thrust due to water on each face of the cube.

2. A rectangular tank 2 metres long, 1·6 metres wide and 1 metre deep contains water to a depth of 8 cm. Calculate the thrust on the 2 m.$\times$1 m. vertical face due to water.

3. A hole, six inches square, is made in a ship's bottom 20 ft. below the water line. What force must be exerted to keep the water out by holding a piece of wood against the hole ? (1 cu. ft. of sea water weighs 64 lb.)

4. The flat bottom of a vessel is moveable, and has an area of 56 sq. cm. The vessel contains mercury to a height of 5 cm. and above the mercury water to an additional height of 32 cm. Calculate the upward force necessary to keep the bottom in position, given that the specific gravity of mercury is 13·6.

5. A cylinder, sufficiently long to ensure that the upper end is in air, is immersed vertically in sea water (sp. gr. 1·02). The lower end of the cylinder is closed by a brass plate (sp. gr. 8·6) 1 cm. thick and of the same diameter as the cylinder. How far

below the surface of the sea water must the plate be immersed in order that the fluid pressure will just hold the plate against the cylinder?

6. A square is placed in a heavy liquid with one side in the surface. Show how to draw a horizontal line in the square dividing it into two parts, the thrusts on which are equal.

7. A parallelogram $ABCD$ is immersed vertically in a liquid with the side AB in the surface. Show how to draw a line from B dividing the parallelogram into two parts, the thrusts on which are equal.

8. A rectangular area is immersed in a heavy liquid with two sides horizontal, and is divided by horizontal lines into strips on which the total thrusts are equal. Prove that, if a, b, c are the breadths of three consecutive strips,

$$a(a+b)(b-c)=c(b+c)(a-b).$$

9. Find the thrust on a vertical quadrilateral which has one side of length a in the surface, and the opposite side of length b parallel to it at depth h.

10. A cone, full of water, is placed on its side on a horizontal table. Show that the thrust on its base is $3 \sin \alpha$ times the weight of the contained fluid, where α is the semi-vertical angle of the cone.

11. An open circular cylinder, 3 feet long with circular base of radius 5 inches, contains water to a depth of 2 feet. It is tilted about a point on the rim of its base until the water is on the point of running out. Find the thrust on the base. [*Banaras*, 1955]

12. A rectangular box is half filled with water and the remaining half with oil. If the oil be half as heavy as water, show that the total thrust on a side of the box is one fourth greater than it would be, if the box were filled with oil only.

13. A flat triangular plate is immersed in water in a vertical position. The base is horizontal and 6 feet below the surface. The vertex is 2 feet below the surface. Find the depth of the centre of pressure. [*Banaras*, 1957]

14. Find the centre of pressure of a vertical circular area of radius a immersed with its centre of gravity at a depth h below the free surface. [*Roorkee*, 1961]

15. A rhombus is immersed in a liquid with vertex in the surface and the diagonal through the vertex vertical. Prove that the centre of pressure divides the diagonal in the ratio 7 :5. [*Banaras*, 1963]

16. Prove that the depth of the centre of pressure of a trapezium immersed in water with the side a in the surface and the parallel side b at a depth c below the surface is

$$\left(\frac{a+3b}{a+2b}\right)\frac{c}{2}.$$ [*Banaras*, 1964]

17. A quadrilateral is immersed vertically having two sides of length $2a$, a parallel to the surface at depths h, $2h$ respectively. Show that the depth of the centre of pressure is $\frac{3}{2}h$.

18. Find the position of the centre of pressure on a circular gate, 4m. in diameter placed with its centre 4m. below the water surface and in a plane inclined at 45° to the vertical.

19. An ellipse is just immersed in a fluid with its major axis vertical. Show that if the centre of pressure coincides with a focus, the eccentricity of the ellipse is $\frac{1}{4}$.

20. In the vertical side of a vessel containing water there is a square trap door, opening freely outwards about a hinge in its upper edge, two sides of the square being horizontal. The length of the side is 3 cm., and the depth of the hinge below the surface of the water is 9 cm. Find the least force (in grams) that will keep the trap door closed.

21. A tunnel, of rectangular section, of height h, is closed by a heavy uniform metal door, inclined at an angle α to the vertical, and swung on hinges along the roof of the tunnel. Show that if the door is to open automatically just when the level of water in the tunnel rises to the roof, the weight per unit area of the door must be $\frac{2}{3}wh$ cosesc α, where w is the weight of unit volume of water.

22. A circular flap, 2 feet in diameter, is used to close a hole in the side of a tank; it is kept in place by bolts at the highest and lowest points of the flap. Calculate the forces on these bolts when the water is 5 feet above the top of the flap, taking the weight of a cubic foot of water to be 62·5 lb.

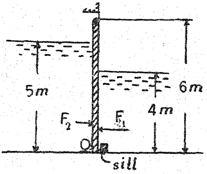
11·9. Water Gates and Dams. The applications of fluid pressure in engineering practice are found in water gates and dams. A water gate is a gate put in a canal to regulate the flow of water. The water level in the canal may be different on the two sides of a water gate. A water gate is of sturdy construction and is generally made of iron. In some cases the water gate opens in two parts about vertical hinges at its sides. In other cases, it can slide up and down in vertical guides.

A dam is an earth or concrete structure designed to prevent a large expanse of water escaping away. A dam can be regarded as a rigid body, free to slide as a whole along the surface on which it rests, or to turn about an edge under the action of the fluid pressure on one side of the dam. A dam should be made sufficiently heavy and broad based so as to withstand both sliding and overturning.

The following examples illustrate some of the engineering applications of fluid pressure.

Ex. 1. A rectangular lock gate, 3 metres wide and 6 metres high, is hinged at the upper 3-metre edge and rests against a sill at the lower edge. Water stands 4 metres high on the sill side and 5 metres high on the other side above the bottom of the gate. Find the magnitude and position of the resultant thrust, the force on the sill and the reaction at the hinge.

The centre of pressure for a rectangle just immersed in water, is $\frac{2}{3}h$ from the surface of the water, or $\frac{1}{3}h$ from the bottom, where h is the height of the rectangle.



The thrust F_1 on the lock gate due to the water on the sill side in given by

$$F_1 = w\bar{z}_1 A_1 = w \times 2 \times 4 \times 3$$
$$= 24 \text{ metric tonnes,}$$

since 1 cubic metre of water weighs one metric tonne. F_1 acts at $\frac{1}{3}h_1$, i.e. $\frac{4}{3}$ metres above the bottom O of the gate.

The thrust on the other side is

$$F_2 = w\bar{z}_2 A_2 = w \times \tfrac{5}{2} \times 5 \times 3$$
$$= 37{\cdot}5 \text{ metric tonnes,}$$

and acts at $\frac{1}{3}h_2$, i.e. $\frac{5}{3}$ metres above O.

Therefore the resultant thrust R

$$= F_2 - F_1 = 37{\cdot}5 - 24$$
$$= 13{\cdot}5 \text{ metric tonnes.}$$

Let x be the height above O of the line of action of R, then taking moments about O, we have

$$Rx = F_2 \,.\, \tfrac{5}{3} - F_1 \,.\, \tfrac{4}{3}.$$

or $$13{\cdot}5x = 37{\cdot}5 \times \tfrac{5}{3} - 24 \times \tfrac{4}{3},$$

whence $$x = \tfrac{61}{27} \text{ metres } = 2{\cdot}259 \text{ metres.}$$

Let R_1 and R_2 be the reaction at the hinge and the force on the sill respectively, then

$$R_1 + R_2 = R = 13{\cdot}5.$$

Taking moments about O.

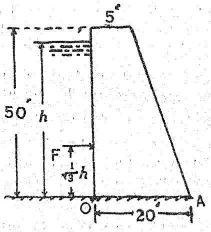
$$R_1 \cdot 6 = Rx = 13{\cdot}5 \times \tfrac{61}{27}.$$

Therefore $$R_1 = \frac{13{\cdot}5 \times 61}{6 \times 27} = 5{\cdot}08 \text{ metric tonnes.}$$

$$R_2 = 13{\cdot}5 - 5{\cdot}08 = 8{\cdot}42 \text{ metric tonnes.}$$

Ex. 2. The vertical cross-section of a concrete dam, 50 feet high, is a trapezium, 5 feet wide at the top and 20 feet wide at the bottom, as shown in the figure.

The material of the dam weighs 150 pounds per cubic foot. How high can water stand against the vertical face without overturning the dam? If the coefficient of friction between the material of the dam and the surface on which it rests be 0·8, how high can water stand without causing the dam to slide? One cubic foot of water weighs 62·5 pounds.



Consider a vertical slice of the dam, one foot long. Let the water stand h feet high.

The thrust on this slice is,

$$F = w\bar{z}A = 62{\cdot}5 \times \frac{h}{2} \times 1 \times h$$

$$= 31{\cdot}25h^2 \text{ lb.}, \qquad \ldots \quad (1)$$

and it acts at $\frac{1}{3}h$ feet above the base of the dam.

The weight of the slice is

$$W = \tfrac{1}{2}(5+20)50 \times 150$$

$$= 625 \times 150 \text{ lb.} \qquad \ldots \quad (2)$$

The distance of centre of gravity of the section from the vertical face is

$$\bar{x}=\frac{5\times50\times5/2+\frac{1}{2}\times50\times15(5+5)}{\frac{1}{2}(5+20)\times50}=7 \text{ ft.}$$

When the dam is on the point of turning over about the outer edge A, the total reaction on the dam of the earth's surface acts at A. Taking moments about the edge A, we have

$$F\times\tfrac{1}{3}h=W(20-\bar{x}),$$

or $$31{\cdot}25h^2\times\tfrac{1}{3}h=625\times150\times13,$$

or $$h^3=60\times150\times13,$$

or $$h=48{\cdot}9 \text{ ft.}$$

If the dam is on the point of sliding, the force of water is balanced by the limiting frictional resistance,

i.e. $$F={\cdot}8R={\cdot}8W,$$

or $$31{\cdot}25h^2={\cdot}8\times625\times150,$$

or $$h^2=16\times150,$$

or $$h=49 \text{ ft.}$$

Examples 20

1. A rectangular lock gate is 10 feet wide. The height of water above the bottom of the gate on one side is 8 feet and on the other side 6 feet. Calculate in pounds weight the thrust on each side of the gate and indicate in each case the height of the centre of pressure. What is the resultant thrust on the gate and at what height above the bottom does it act? (Weight of 1 cu. ft. of water is 62·5 lb.) [*Banaras*, 1954]

2. A sluice gate, when closed has water on both sides. The width of the gate is 10 metres. The level of water on one side of it is 6·4 metres and on the other side 4·8 metres from the base of the gate. Calculate the resultant thrust and the distance, measured from the base, where it acts.

3. The flood gate on a power dam is a rectangle 10 feet high and 12 feet long, and can slide up and down on four rollers, one at each corner of the gate. Calculate the reaction on each roller when the upper edge of the gate is 15 feet below the surface of water. [*Banaras*, 1961]

4. A vertical sluice gate, 3 m. high and 9 m. long, weighs 8000 kg. If the coefficient of friction between the gate and the slots in which it slides is 0·25, what lifting force would be needed to open it when water stands even with its top on one side?

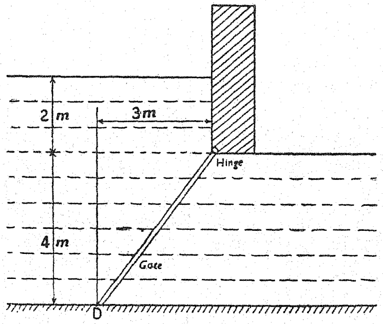
What lifting force would be required to raise a submerged gate having the same dimensions as that of the above problem if its top is 3 m. below the water surface?

5. The gate of a lock is 10 ft. wide and 18 ft. deep and it has the pressure due to 15 ft. of fresh water on one side and 10 ft. of sea water on the other side. Find the magnitude and the position of the resultant thrust on the surface of the gate. (Fresh water weighs 62·5 lb. and sea water 64 lb. per cu. ft.)

[*Banaras*, 1962]

6. Two lock gates, each 5 metres wide, are closed across a canal, so that each makes an angle of 30° with the line joining their hinges. The depth of water on one side is 4 metres and on the other 2 metres. Find the force with which the two gates press on each other.

7. The gate, a cross-section of which is shown in the figure, is 2 m. wide. Find the thrust and the distance of the centre of pressure from the hinge (i) on the right surface of the gate; (ii) on the left surface of the gate. (iii) Find the vertical force at *D* required to open the gate, neglecting its weight.



8. The dam of a reservoir is 200 metres long and its face towards the water is rectangular and inclined at 30° to the horizon. Find the thrust acting on the dam when the water is 10 metres deep.

9. The bed of a canal is a horizontal plane and its banks are planes, each inclined at 45° to the bed, 20 feet apart at the bottom. Calculate the thrust on a length of 1 foot of the bed of the canal due to a depth of 8 feet of water. Compare this with the weight of water in the canal per foot length. Account for the difference.

[*Banaras*, 1958]

10. A vertical masonry dam is in the form of a trapezium 300 feet long at the top, 100 feet long at the bottom and 230 feet high. What pressure must it withstand when the water stands up to the top? Locate the position of the resultant force.

11. A gravity dam 9 metres high has a trapezoidal cross-section 3 metres wide at the top and the side facing the water vertical. If the dam be made of concrete of specific gravity 2·5, find the minimum width b at the base for which the dam will have a factor of saftey 2 against overturning when the water stands up to the top.

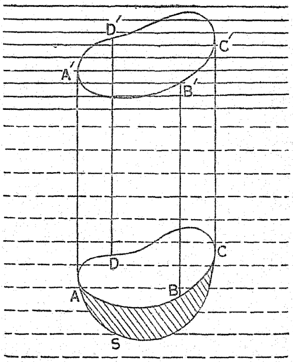
[Hint. Factor of saftey is the ratio of the thrust which would just overturn the dam to the thrust which actually acts on the dam. Thus the moment of the weight of the dam about the outer edge equals double the moment of the thrust.]

CHAPTER XII

THRUSTS ON CURVED SURFACES. FLOATING BODIES

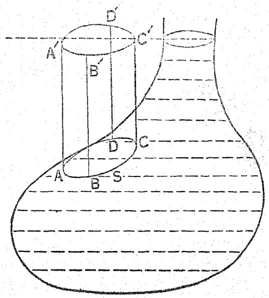
12·1. Vertical Thrust on a Curved Surface. When a curved surface is immersed in a heavy fluid, the thrusts on the various elements of the surface act normal to the element, and so act in different directions. Hence the resultant thrust on the curved surface may not be a single force but may be a force and a couple. Whereas we may not be able to find the couple easily, we can always find the vertical component of the resultant thrust, and its component in any given horizontal direction.

Let S be the curved surface on which the thrust is to be calculated, and let $ABCD$ be the curve bounding it. From every point on this curve draw vertical lines cutting the surface of the fluid in the curve $A'B'C'D'$. These lines form a cylinder. Consider the equilibrium of the fluid enclosed within this cylinder between $A'B'C'D'$ and the surface S. The only vertical forces acting upon this fluid are the weight of the fluid and the vertical component of the reaction exerted by the surface S upon the fluid. Hence these must be equal and opposite to each other. But the



reaction exerted by the surface S on the fluid is equal and opposite to the thrust exerted by the fluid upon S. Therefore the vertical component of the thrust on S is equal to the weight of the fluid enclosed in the cylinder between S and the surface of the fluid. This enclosed fluid is known as the *superincumbent fluid.*

In some cases (e.g. the one shown in the marginal figure), there is fluid below the surface S but no fluid above it. In such a case we still draw the vertical lines from the bounding curve $ABCD$, to cut the plane of the fluid surface in $A'B'C'D'$. The vertical component of the thrust on surface S is equal to the weight of the fluid which would occupy the so formed cylinder from S to $A'B'C'D'$, and will act *upwards*. The reason is that in this case also the pressure at any point of the surface is proportional to the depth of the point below the surface of the fluid, only the liquid is pressing the surface upwards.



Care should be taken in cases where the surface S bends upon itself, so that the fluid is above a part S_1 of the surface and below the remaining part S_2. In such a case the superincumbent fluid for S_1 and S_2 should be found separately. The vertical thrust will be equal to the weight of the difference of the two.

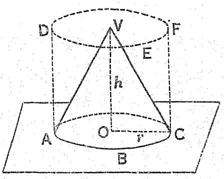
NOTE. If the fluid is a liquid with atmospher above it, then the vertical component of thrust on S is equal to the weight of the superincumbent liquid plus the force due to atmospheric pressure on $A'B'C'D'$.

Ex. A conical wine glass is filled with water and placed in an inverted position upon a table. Show that the thrust of the water upon the glass is two-thirds of that upon the table.

Let the height of the cone be h and the radius of its base r. When the glass is placed in an inverted position on the table, the surface in contact with water is the circle ABC of area πr^2. Its depth below the water surface is h. Hence the thrust on it

$$=\pi r^2 hw,$$

where w is the weight of unit volume of water.



If we draw vertical lines through points on the circle ABC, we get a circular cylinder which cuts the horizontal plane through the vertex V in another circle DEF. The fluid superincumbent on the cone is the fluid which would fill the cylinder $ABCDEF$ above the cone $VABC$. Hence the thrust of water on the wine glass

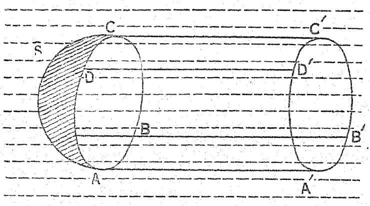
$$\begin{aligned}&=\text{weight of the superincumbent fluid}\\&=w(\text{volume of cylinder}-\text{volume of cone})\\&=w(\pi r^2h-\tfrac{1}{3}\pi r^2h)=\tfrac{2}{3}\pi r^2hw\\&=\tfrac{2}{3}(\text{thrust on the table}).\end{aligned}$$

[The student will note that the weight of the water inside the cone is only one-third of the fluid thrust on the table. The remaining two thirds of the fluid thrust is due to the downward thrust exerted by the wine glass on the water. This, of course, is equal and opposite to the thrust exerted by the water on the wine glass. The wine glass must be of sufficient weight to sustain this thrust, otherwise it will rise and the water will escape from the sides.]

12·2. Horizontal Thrust. *To find the component of thrust in a given horizontal direction on a surface immersed in a heavy fluid.*

Let the immersed surface S be bounded by the curve $ABCD$. Through every point of the bounding curve draw lines parallel to the given horizontal direction. These lines form a cylinder, which cuts a vertical plane perpendicular to the given direction in the curve $A'B'C'D'$.

Consider now the equilibrium of the fluid enclosed



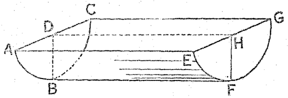
within the cylinder between the surface S and the plane $A'B'C'D'$. The only horizontal forces acting on this fluid mass in the direction AA' are the component of the reaction of S and the thrust of the fluid on $A'B'C'D'$. These must be equal and opposite to each other, and must act in the same straight line. But the reaction of S is equal and opposite the thrust of the fluid on S. Hence the horizontal component of the thrust on the surface S is equal to the fluid thrust on the area $A'B'C'D'$. Since this is a plane area, the thrust on it can be calculated by the method of § 11·7. It should be noted that the area $A'B'C'D'$ is the projection of the surface S on a plane perpendicular to the given direction.

Corollary. *The horizontal thrust in any direction on a closed surface immersed in a heavy fluid, is zero.*

The closed surface can be divided into two parts the projections of which on a plane perpendicular to the given direction are the same. Therefore the horizontal thrusts on the two parts are equal and opposite, and the resultant thrust on the closed surface is zero.

Ex. A horizontal trough is semi-circular in section and is filled with water whose weight is W. If the trough is imagined to be divided into two halves along the middle, show that the water will tend to push them apart horizontally with a force W/π.

Let the trough be $ABCEFG$, of length l and radius a. Then



the weight of the water contained in the trough is

$$W = \tfrac{1}{2}\pi a^2 l w.$$

The horizontal thrust on half the curved surface $ABFE$ will be in the direction CA. The projection of the surface $ABFE$ on a plane perpendicular to CA is $DBFH$. The thrust on this area

$$= al \,.\, \tfrac{1}{2}a \,.\, w = \tfrac{1}{2}a^2 l w = W/\pi.$$

Hence the horizontal thrust on half the curved surface is also equal to W/π.

12·3. Resultant Thrust. By finding the vertical thrust V on a curved surface immersed in a heavy fluid, and the components H_1, H_2, of the thrust in two mutually perpendicular horizontal directions we can find the magnitude of the resultant thrust R by compounding them. Thus

$$R = \sqrt{(V^2 + H_1{}^2 + H_2{}^2)}.$$

The lines of action of these components can also be found. It is easily seen that the vertical component of thrust on a curved surface will act through the centre of gravity of the superincumbent fluid. Similarly, the horizontal component of thrust in any direction will act through the centre of pressure of the projection of the surface on a vertical plane perpendicular to this direction. ($A'B'C'D'$ in the figure on p. 211). If the lines of action of the three components V, H_1 and H_2 meet at a point, the resultant thrust reduces to a single force. If they do not meet at one point the resultant reduces to a force and a couple.

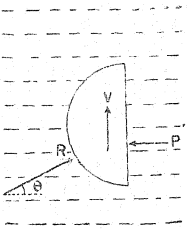
When the curved surface is such that its bounding curve lies in a plane, we can obtain the resultant thrust more simply by the application of Archimedes principle. This is given in the next section.

12·4. Principle of Archimedes. *When a body is wholly or partially immersed in a heavy fluid at rest the resultant thrust of the fluid on the body is equal and opposite to the weight of the fluid displaced by the body, and acts through the centre of gravity of the displaced fluid.*

PROOF. Suppose the body is removed and the space filled with fluid of the same nature as that surrounding the body. Since the pressure at any point in the original fluid depends only on the depth of the point below the surface, it is unaltered by this substitution. Hence the same resultant thrust as was acting on the body, now acts on the fluid replacing the body. But this added fluid, together with the surrounding fluid, would form one continuous mass at rest under gravity. Therefore the resultant thrust is balanced by the external forces acting on the fluid which now fills the space formerly occupied by the body, namely its weight. Hence the resultant thrust on the body is equal and opposite to the weight of the fluid which would fill the space occupied by the body, and acts through its centre of gravity. This amount of fluid is designated by the name "displaced fluid".

The resultant fluid thrust on the body is called the *force of buoyancy*; and the centre of gravity of the displaced fluid, through which this force acts, is called the *centre of buoyancy*.

The following example would show the use of the principle of Archimedes in finding the resultant thrust on a curved surface which is bounded by a plane curve.



Ex. A hemisphere of radius a is immersed in a liquid of density ρ with its plane base vertical and centre at depth $a\sqrt{5}$ below the surface. Show that the resultant force on the curved surface is $\frac{7}{3}\pi\rho g a^3$, and that its direction makes an angle

$$\tan^{-1}(2/\sqrt{45})$$

with the horizontal.

Let P be the thrust on the plane base and R the resultant thrust on the curved surface at an angle θ to the horizontal.

The resultant of these two forces is the upthrust V, which by Archimedes' principle is equal to the weight of fluid displaced by the hemisphere. Thus

$$V=\tfrac{2}{3}\pi\rho g a^3.$$

Also
$$P=\pi a^2 \,.\, \rho g \,.\, a\sqrt{5}.$$

Therefore
$$R\cos\theta=P=\pi\sqrt{5}\rho g a^3,$$

and
$$R\sin\theta=V=\tfrac{2}{3}\pi\rho g a^3.$$

Hence
$$R^2=(5+\tfrac{4}{9})(\pi\rho g a^3)^2,$$

or
$$R=\tfrac{7}{3}\pi\rho g a^3;$$

and
$$\tan\theta=\frac{2}{3\sqrt{5}}=\frac{2}{\sqrt{45}}.$$

EXAMPLES 21

1. A hollow cylinder closed at both ends is filled with water and held with its axis horizontal. Find the vertical thrust on the lower half of the curved surface.

2. A hemispherical bowl filled with water is inverted and placed with its plane base in contact with a horizontal table. Prove that the vertical thrust on the bowl is one-third of the thrust on the table.

3. A hollow cone of height h, closed by a base of radius r, is filled with water and held with its axis horizontal. Find the resultant vertical thrust on (i) the upper half, (ii) the lower half, of the curved surface.

4. A heavy conical cup is placed with its vertex upwards on a smooth horizontal plane, and water is gradually poured in through a hole in the top. If the weight of the cup is 5/8ths of the weight of water which would just fill it, prove that the cup will be on the point of rising from the plane when the water has reached half the height of the cup.

5. A double funnel is formed by joining two equal hollow cones at their vertices and stands on a horizontal table with the common axis vertical. Liquid is poured into the funnel until its surface bisects the axis of the upper cone. If the liquid be on the point of escaping between the lower cone and the table, prove that the weight of either cone is to that of the liquid which it can hold as 27 :16.

6. A conical vessel contains enough fluid to fill it to a depth equal to half the depth of the vessel. If the vessel be inverted on a horizontal table and no fluid be allowed to escape, find the ratio in which the resultant thrust on the curved surface of the vessel is altered. [*Roorkee*, 1961]

7. A solid cone of height h and radius of base r, is just immersed with its axis vertical and vertex downwards in a liquid of density ρ. Find the resultant horizontal thrust on half of the curved surface cut off by a plane through the axis.

8. A hollow right circular cone filled with liquid is held with its axis vertical and vertex downwards. Find the magnitude and the line of action of the resultant fluid thrust on half the surface of the cone cut off by a vertical plane through its axis. [*Roorkee*, 1963]

9. A hollow right circular cylinder is filled with liquid and held with its axis horizontal. Find the magnitude and the line of action of the resultant thrust on half the curved surface cut off by a vertical plane through the axis.

10. A hemispherical bowl is filled with water. Find the horizontal fluid pressure on one half of the surface divided by a vertical diametral plane, and show that it is $1/\pi$ times the resultant fluid thrust on the whole surface. [*Roorkee*, 1962]

11. The end of a horizontal pipe is closed by a sphere of the same radius as the internal section of the pipe. The sphere is hinged at the highest point. If the pipe is just full of liquid of density ρ, prove that the moment about the hinge of the liquid pressure on the sphere is $g\rho\pi a^4$.

12. Removable gates used for river control frequently have the form of cylindrical drums. Determine the magnitude and line of action of the hydrostatic force on a drum gate 2·5 metres in diameter and 9 metres in length when the water level is at the top of the gate. [*Roorkee*, 1964]

13. A spherical shell formed of two halves in contact along a vertical plane is filled with water. Show that the resultant thrust on either half of the shell is $\frac{1}{4}\sqrt{13}$ of the total weight of the liquid.

14. A solid sphere of density ρ is placed at the bottom of a vessel which is horizontal, and a liquid of density $\sigma(<\rho)$ is poured in so as just to cover up the sphere. The sphere is then cut along a vertical diametral plane. Prove that the two parts will not separate if $4\sigma \geqslant \rho$.

15. A solid hemisphere is immersed in a liquid with the highest point of its plane base in the surface, and the base inclined at $\tan^{-1} 2$ to the horizon. Show that the resultant thrust on the curved surface is equal to twice the weight of the displaced liquid.

16. A right circular cone is filled with water; it is then closed and laid on a table with a generating line in contact with it. Find the resultant vertical and horizontal thrusts upon the curved surface.

12·5. Equilibrium of a floating body. When a body floats freely in a fluid the only forces acting on it are its weight and the resultant thrust of the fluid on it. For equilibrium these two must be equal and opposite to each other and must act in the same straight line. Since by the principle of Archimedes, the latter force is equal and opposite to the weight of the fluid displaced by the body, it follows that in equilibrium the weight of a floating body is equal to the weight of the fluid displaced by it, and that the centres of gravity of the body and the displaced fluid are in the same vertical line.

If ρ is the density of the floating body and V its volume, its weight is $\rho g V$. Let ρ' be the density of the fluid and V' the volume of the displaced fluid. Then $\rho' g V'$ is the weight of the displaced fluid. Hence, for equilibrium,

$$\rho' g V' = \rho g V,$$

or

$$V' = \frac{\rho}{\rho'} V.$$

Since $V' \leqslant V$, the solid will float in the fluid only if $\rho \leqslant \rho'$.

When a body floats partly immersed in one fluid and partly immersed in another, we see by a similar reasoning that for equilibrium the weight of the body must be equal to the combined weight of the displaced fluids, and must act through the resultant centre of gravity of the whole displaced fluid.

12·6. Body floating under constraint. When a floating body is free to turn about a fixed point, there are three forces acting on the body. These are

(i) the weight of the body acting through its centre of gravity G.

(ii) the force of buoyancy acting through the centre of gravity G' of the displaced fluid and

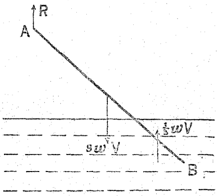
(iii) the reaction at the fixed point O.

Since the first two forces are vertical, for equilibrium the third force also must be vertical and the points O, G, G' must be in the same vertical plane. Taking moments of the forces about O, will generally give the condition of equilibrium.

We can similarly consider the equi ibrium of a floating body suspended by a string attached to one of its points. In this case the third force will be the tension of the string, and equilibrium will be obtained only when the string is vertical.

Ex. A uniform rod capable of turning about one of its ends, which is out of water, rests inclined to the vertical position with one third of its length in water. Prove that its specific gravity is 5/9.
[*Roorkee*, 1963]

Let s be the specific gravity of the rod and V its volume. Then the volume $\frac{1}{3}V$ is immersed in water. Let w be the weight of unit volume of water; then the forces acting on the rod are its weight swV, the force of buoyancy $\frac{1}{3}wV$, and the reaction R acting at the end A. If the length of the rod is $6a$, the weight acts at distance $3a$ from A, and the force of buoyancy at distance $\frac{1}{2}(\frac{1}{3}\times 6a)$ from B, i.e. $5a$ from A.



Taking moments about A, we have

$$swV \,.\, 3a = \tfrac{1}{3}wV \,.\, 5a,$$

or $$s = \tfrac{5}{9}.$$

EXAMPLES 22

1. A hollow cylindrical vessel, one end open, diameter 100 cm., floats vertically, with closed end uppermost, in water. Neglecting the thickness of the walls, find the weight of the vessel if the level of water outside is 30 cm. higher than the level inside.

2. A tank with vertical sides is 1 m. square in cross-section and 3 m. deep, and is filled to a depth of 2 m. with water. By how much, if at all, will the pressure on one side of the tank be changed if a cube of wood, specific gravity 0·5, measuring 40 cm; on an edge be placed in the water so as to float with one face horizontal ?

3. A solid cone has its axis of length h and is of density ρ; if it floats in a liquid of density $\sigma(>\rho)$, find how much of its axis is out of the liquid when the vertex is (i) upwards, (ii) downwards. [*Roorkee*, 1954]

4. A man whose weight is 80 kg. and whose specific gravity is 1·1, can just float in fresh water with his head above the surface by the aid of a piece of cork which is wholly immersed. Having given that the volume of his head is one-sixteenth of his whole volume and that the specific gravity of cork is 0·24, find the volume of the cork. [*Banaras*, 1962]

5. A balloon of volume V contains a gas whose density is to that of air at the earth's surface as 1 : 15. If the envelope of the balloon be of weight w but of negligible volume and the density of the air be σ find the acceleration with which it will begin to ascend. [*Roorkee*, 1962]

6. A cone of radius a and density ρ, floats with its axis horizontal in a liquid of density double its own. Find the thrust on its base and prove that, it θ be the inclination to the vertical of the resultant thrust of the curved surface, and α, the semi-vertical angle of the cone, then $\tan\theta=(4/\pi)\tan\alpha$.

7. A solid circular cone of uniform material and of height h and semi-vertical angle α, floats in water with its axis vertical and vertex downwards and a length h' of its axis immersed. The cone is bisected by a vertical plane through its axis and the two parts are hinged together at the vertex. Show that the two parts will remain in contact if $h' \geqslant h \sin^2\alpha$.

8. A rectangular block of wood 40 cm. in depth and of specific gravity 0·9, floats in water with its upper face horizontal. Oil of specific gravity 0·6 is poured on the water till the block is completely immersed. Show that the wood will rise through 6 cm.

9. A right circular cone of density ρ floats just immersed with its vertex downwards in a vessel containing two liquids of densities σ_1 and σ_2 $(\sigma_1>\sigma_2)$. Show that the plane of separation of the two liquids cuts off from the axis of the cone a fraction

$$\{(\rho-\sigma_2)/(\sigma_1-\sigma_2)\}^{1/3} \text{ of its length.}$$

10. A ship sailing from the sea into a river sinks through a distance b, and on unloading a cargo of weight P there, rises through a distance c. Show that the weight of the ship after unloading is

$$\left[\frac{b\sigma}{c(\sigma-\rho)}-1\right]P,$$

where σ and ρ are the densities of sea and river water respectively. The sides of the ship are assumed to be vertical at the water level.

11. A steamer in going from salt water into fresh water was observed to sink 5 cm. but, after burning 50 tonnes of coal, to rise 2·5 cm. Supposing the densities of salt and fresh water to be as 65 : 64, find initial displacement of the steamer in tonnes.

[*Banaras*, 1963]

12. A body floating in water has volumes v_1, v_2, v_3 above the surface when the densities of surrounding air are respectively ρ_1, ρ_2, ρ_3. Prove that

$$\frac{\rho_2-\rho_3}{v_1}+\frac{\rho_3-\rho_1}{v_2}+\frac{\rho_1-\rho_2}{v_3}=0.$$

13. A solid hemisphere floats completely immersed with a point of the rim joined to a fixed point by means of a string. Show that the base is inclined to the vertical at an angle $\tan^{-1}(3/8)$.

[*Roorkee*, 1962]

14. A rod of small section and density m has a small piece of lead of weight $1/n$th that of the rod attached to one extremity. Prove that the rod will float at any inclination in a liquid of density m' if

$$(n+1)^2 m = n^2 m'. \qquad [\textit{Banaras}, 1964]$$

15. A uniform rod, of length $2a$, floats partly immersed in a liquid, being supported by a string fastened to one of its ends. If the density of the liquid be $\frac{4}{3}$ times that of the rod, prove that the rod will rest with half its length out of the liquid. Find also the tension of the string.

16. A uniform rod of length $2a$ can turn freely about one end which is fixed at a height $h\ (<2a)$ above the surface of a liquid. If the densities of the rod and the liquid be ρ and σ, show that the rod can rest either in a vertical position or inclined at an angle θ to the vertical, where

$$\cos\theta=\frac{h}{2a}\sqrt{\left(\frac{\sigma}{-}\right)}.$$

ANSWERS

Examples 1 (Pages 12-13)

1. 340 kg., 668 kg. **2.** 56·6 N., 18·4 N.

Examples 2 (Pages 19-21)

5. (*a*) 119·5 kg., 74° 3′; (*b*) 3675 N. 282° 7′.

6. $Q=75$ kg., in direction BA; $R=125$ kg.

7. 14·43 kg., 28·87 kg. **8.** 377 kg., 42° 18′ north of east.

11. (*a*) 3,000 kg.-cm., (*b*) 57·73 kg., (*c*) 50 kg.

Examples 3 (Pages 29-32)

1. $W\sqrt{13}/2\sqrt{3}$, $W/2\sqrt{3}$. **2.** 120 kg.

3. 12·4 kg., 5·3 kg. **7.** 60·4 kg.

8. 149 lb., perpendicular to radius joining stone and centre.

9. $\frac{2}{3}W\sqrt{3}$, $\frac{1}{3}W\sqrt{3}$. **10.** 1470 kg., 3020 kg.

11. $74\frac{2}{3}$ lb.

12. 546·4 kg., 282·8 kg., 386·4 kg., 546·4 kg.

13. 10·4 kg., 12 kg., 53° 3′. **15.** 2247 kg., 2894 kg.

17. 4·33 kg., 6·62 kg., 57·32 cm.

18. 173 kg. **19.** 3·5 tonnes, compressive; 0.

Examples 4 (Pages 41-43)

1. 8 kg., 6 kg.

3. (*a*) 20 kg. at 1·75 m.
(*b*) 100 kg. at 0·4 m. (*c*) 140 kg. at 0 m.

5. 10 kg.; 33·3 cm., 16·7 cm. **6.** 4 m. from weaker man.

7. 945 kg. **8.** 2′ 5″ from B.

9. $\frac{1}{2}W+n(n-1)aw/2l$, $\frac{1}{2}W+\{2l-(n-1)a\}nw/2l$.

10. 2·37 m. or 5·63 m. **11.** 477·5 kg., 572·5 kg.

12. 6 m. **13.** $5\frac{1}{3}$ ft-lb., clockwise.

14. 3 inches. **16.** 80° 57′; 3·29 kg.

17. $\sin^{-1}(2G/lW)$. **19.** $\frac{9}{8}l$; w, $4w$.

20. 360 lb.

Examples 5 (Pages 48-50)

1. (*b*) 41·55 kg., 69° 42′; $39{\cdot}5\,x-12{\cdot}9y=58{\cdot}8$.

2. 7·7 lb. along AD and 37·7 lb. parallel to BA at 4·66 inch from A opposite to D.

4. $3\sqrt{3}$ kg., 270°; at $\frac{1}{3}BC$ from B.

5. $2\sqrt{3}\ P$, 30°; cutting AB at $\frac{7}{3}AB$ from A.

6. 50·9 kg., 162° 52′; divides BC in ratio 2 : 1.

8. $\sqrt{\{P^2+Q^2+R^2-\sqrt{2}R(P+Q)\}}$, $P(x+y-1)-Q(x-y-1) = \sqrt{2}Ry$.

9. $2\sqrt{2}P$, cutting AB at $-\frac{3}{2}a$ from A and BC at $\frac{3}{2}a$ from B.

11. At $3a$ (a = side) from the first corner.

12. (5·04 kg., 240° 42′), 12·25 m.

Examples 6 (Pages 54-58)

1. 144 kg.; 255·2 kg., 70° 24′. **2.** $10\sqrt{3}$ kg., $30\sqrt{3}$ kg.

3. $\frac{1}{2}W$.

4. $B_x = D_x = C_x = 385$ kg., $B_y = D_y = 2000$ kg., $C_y = 0$.

9. 6·6 lb., 30·7 lb., 77°. **10.** 121·8 kg., 137° 40′; 90 kg.,

13. 159 lb., 124 lb., 124 lb.

15. $2W \sec \alpha$, $Wr/\sqrt{(2ar-r^2)}$, $W(a-r)/\sqrt{(2ar-r^2)}$, $W(a-r)/\sqrt{(2ar-r^2)}-2W \tan \alpha$.

16. 6·4 kg., 5 386 kg., 5 kg., 5·386 kg., 6·4 kg.

17. 7·962 lb. wt. at 144° 43′ to BC; 6·196 BC lb.-ft.

18. 15·7 cwt. **19.** 884 lb.; 625 lb., 1625 lb.

20. 500 lb., 4950 lb., 4030 lb. **23.** a, $2\sqrt{3}\ W$.

Examples 7 (Pages 77-79)

1. —8·3 kg., 33·3 kg., **2.** 389 lb., 349 lb. at 30° to vertical.

In the answers to q. 3, 6, 7 and 12, the stresses in the members specified are given in tons; and those in q. 5 and 9 in metric tonnes.

3. $AC = 15·0$ (C), $AB = 7·0$ (T), $BC = 8·0$ (C), $BD = 13·1$ (T).

4. $AC = 1100$ kg. (C), $CB = 1345$ kg. (C), $CD = 1500$ kg. (T), $BD = AD = 950$ kg. (T).

5. $AB = DE = 8·5$ (C), $BC = CD = 5·7$ (C), $BF = DF = 2·8$ (C), $AF = EF = 6$ (T), $CF = 4$ (T).

6. $AB = DE = 46·2$ (C), $BD = 34·6$ (C), $BC = DC = 23·1$ (T), $AC = CE = 23·1$ (T).

7. $AB = BC = 5$ (C), $CD = CE = 6$ (C), $DE = 8·5$ (T), $CF = 1·4$ (C), $AF = 7·1$ (T), $BF = 4$ (C), $EF = 6$ (T).

8. $AB = BC = CD = DE = 2310$ kg. (C), $BG = DF = GF = 2310$ kg. (T), $AG = FE = 1150$ kg. (T), $CG = CF = 0$.

9. $AB = 22·5$ (T), $EF = 3·5$ (C), $BG = 17·3$ (C), $HG = 31·2$ (C).

10. $1=3=5=773$ lb. (T), $2=4=6=773$ lb. (C), $7=8=9=10=0$.

11. $AB=4620$ kg. (T), $BC=BD=1150$ kg. (T), $CD=2310$ kg. (C), $DE=2000$ kg. (C), $BE=5750$ kg. (C).

12. $AB=BC=9{\cdot}5$ tons (T), $CE=10{\cdot}9$ tons (T), $AE=9{\cdot}2$ tons (C), $BE=4$ tons (C), $ED=16{\cdot}7$ tons (C).

Examples 8 (Pages 88-91)

1. 0·192.
3. (*a*) Slides, (*b*) turns over.
7. 0·273.
8. 58° 38′.
11. 0·0577
14. The other ladder.
15. 4·36 m.

Examples 9 (Pages 97-99)

1. 259·8 kg.
2. 0·1155.
3. 72·3 lb., 38·3 lb.
4. Topples; 18° 26′.
11. 16·28 kg.
13. 19/33; 17° 30′.
14. μP.
15. 3800 lb.
16. 27 lb.
17. 879 lb.
18. 809·8 kg., 158·4 kg.

Examples 10 (Pages 103-105)

1. 1757 kg., 142·3 kg.
2. 1755 kg. 364·8 kg.
3. 2·41 turns.
4. 527·5 lb.
5. 0·238.
6. 2250 lb.
7. 100 lb., 0·29.
8. 406 lb.
9. 326 lb. ft.

Examples 12 (Pages 124-125)

2. 3·54 in.
5. Unstable position when rod is inclined at $\cot^{-1}(2\tan\alpha)$ to horizon.
8. $\angle CAB=60°$, unstable; 180°, stable.

Examples 13 (Pages 135-137)

1. 150 tons.
2. 562·5 t., 605·8 t.
3. 21·47 tonnes., 41·61 m.
4. 70 tons, 78 tons, 344 tons.
5. 8331 kg., 8891 kg., 102·74 m.
6. 187·5 kg.
7. 22·36 times the weight of tape.
8. 106·7 kg.
9. $a\sinh^{-1}(l/a)$; $\sqrt{(l^2+a^2)}-a$.
10. 99·26 m., 5·27 m.
11. 555·75 ft.
13. 25·9 cm.
14. $y=146{\cdot}2\cosh(x/146{\cdot}2)$; (97·6, 180), (63, 160).

Examples 14 (Pages 147-149)

238 kg./cm.^{2} **2.** 0·84 cm.
0·0265 in. **4.** 2·21 tonnes, 0·200 cm.
5·02 tons, 0·019 in **6.** 314 kg./cm.^{2}, 2815 kg.
179·5 H.P. **8.** 4·375.
4·73 cm., 54%. **10.** 3·5 cm.
3·36″. **12.** 14725 kg.-cm., 1·16 cm.
1051 H.P., 0·206°. **14.** 13·97″. **15.** 9·94″, 4·97″.

Examples 15 (Pages 162-163)

20·8 cm.$\times$41·6 cm. **2.** 225 kg./cm.^{2}
1200 lb./in.^{2} throughout the range $2 \leqslant x \leqslant 12$.
153·6 kg./m. **6.** 1008 lb./in.^{2}
12·5 cm. **8.** 22·6 ton-ft. at $x=6·1$ ft. S.F.$=0$.
(i) $Wx^2(3l-x)/6\,EI$, (ii) $wx^2(6l^2-4lx+x^2)/24\,EI$.

Examples 16 (Pages 168-170)

74·16 kg.; 61° 32′, 55° 6′, 48° 8′. **2.** 147·3 kg. **4.** 2 lb.
29·3 lb., 29·3 lb., 41·4 lb. **6.** 40·8 kg.
$8\frac{1}{3}$ tons, $3\frac{1}{3}$ tons. **8.** 1·366 t., 1·414 t., 1·932 t.
221 lb., 442 lb., 552·5 lb.

Examples 17 (Pages 182-184)

150·4 kg.-m., (0·0332, 0·9973, 0·0665).
Force$=0$, Couple$=3320$ lb.-ft. making angles 31° 7′, 112° 20′, 69° 32′ with axes.
30 kg. at $(15, -9)$. **4.** $63\frac{1}{3}$ kg., $38\frac{1}{3}$kg., $38\frac{1}{3}$ kg.
$(\frac{6}{11}a, \frac{10}{11}a)$, a being length of a side.
27 kg., 44 kg., 44 kg. **7.** 50·8 kg., 38·3 kg., 30·8 kg.
64 lb. **9.** 200 lb.
$R=100$ lb., $M=-24$ lb.-ft., axis (53° 8′, 36° 52′, 90°); pitch$=$ 0·24.
60 kg., $R_A=60$ kg. vertical, $R_B=60$ kg. (45°, 45°, 90°).
75 lb., $R_A=61·2$ lb., $R_B=25$ lb.
1750 lb., 2100 lb., 991·7 lb. **15.** $\tan^{-1}\{\mu a/\sqrt{(l^2-a^2)}\}$.

Examples 18 (Pages 194-195)

100 kg. wt. per sq. cm. **2.** 4·854 km.
13 m. **4.** 98 ft.
2·344 lb. wt./sq. in. **6.** $1·066\times10^6$ dynes/sq. cm.
2·66 cm. **10.** $1·530\times10^4$ dynes/sq. cm.

11. 16 cm.; 20·32 cm. **12.** $156\frac{1}{4}$ kg.

13. 2·5 kg. wt. **14.** 7 : 1.

Examples 19 (Pages 200-202)

1. 67·5 kg. wt. on upper face, 94·5 kg. wt. on lower face, and 81 kg. wt. on the remaining faces.

2. 6·4 kg. wt. **3.** 320 lb. wt. **4.** 5·6 kg. wt. **5.** 7·43 cm.

6. At depth $a/\sqrt{2}$, where a is the side of the square.

7. Line meets CD in E where $CE=\frac{3}{4}\,CD$.

9. $\frac{1}{6}(a+2b)h^2w$. **11.** 26·2 lb. wt.

13. $4\frac{6}{7}$ ft. **14.** $h+a^2/4h$.

18. 0·18 m. below the centre, measured along the slope.

20. 49·5 gm. wt.

22. 564·5 lb. wt. at the upper bolt and 613·6 lb. wt. at the lower.

Examples 20 (Pages 205-207)

1. 20,000 lb. wt. $2\frac{2}{3}$ ft.; 11,250 lb. wt., 2 ft.; 8,750 lb. wt., $3\frac{11}{21}$ ft.

2. 89·6 metric tonnes, 2·819 m. **3.** 34,375 lb., 40,625 lb.

4. 18·125, 38·375 metric tonnes **5.** 38,313 lb.wt.,at 6·39 ft. [from bottom.

6. 30 metric tonnes.

7. (i) 20 tonnes, 3·3 m.; (ii) 40 tonnes, 2·9 m.; (iii) 16·7 tonnes.

8. 20,000 metric tonnes. **9.** 10,000 lb. : 14,000 lb.

10. 123,000 ton wt., 138 ft. **11.** 4·76 m.

Examples 21 (Pages 214-215)

1. $(\frac{1}{2}\pi+2)r^2hw$. **3.** $(1-\frac{1}{6}\pi)r^2hw$, $(1+\frac{1}{6}\pi)r^2hw$.

6. 1 : 0·0448. **7.** $\frac{1}{3}rh^2w$.

8. $\frac{1}{6}rhw\sqrt{(\pi^2r^2+4h^2)}$ at $\tan^{-1}(\pi r/2h)$ to horizontal in the plane of symmetry, through a point at distance r/π from axis and at depth $\frac{1}{2}h$.

9. $\frac{1}{4}wr^2h\sqrt{(\pi^2+16)}$ at $\tan^{-1}(\pi/4)$ to horizontal, passing through the centre of the cylinder.

10. $\frac{2}{3}r^3w$. **12.** 35·7 tonnes at 38° 9′ to horizontal, through the centre of the drum.

16. $\pi(\frac{1}{3}+\sin^2 a)hr^2w$, $\pi r^3w\cos^2 a$.

Examples 22 (Pages 217-219)

1. 236 kg. **2.** 64·5 kg.

3. (i) $h(1-\rho/\sigma)^{1/3}$, (ii) $h\{1-(\rho/\sigma)^{1/3}\}$. **4.** 15,550 c.c.

5. $(14\,V\sigma g-15\,w)g/(V\sigma g+15w)$. **6.** $\frac{4}{3}a^3\rho g$.

11. 6500 tonnes. **15.** $\frac{1}{3}$(weight of rod)